Bubbles and Tilings: Art and Mathematics

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Abstract

The 2002 proof of the Double Bubble Conjecture on the ideal shape for a double soap bubble depended for its ideas and explanation on beautiful images of the multitudinous possibilities. Similarly recent results on ideal tilings depend on the artwork.

Bubbles

I'm a geometer. To do mathematics, I have to have a picture in mind. For example, I like to picture soap bubbles. Soap bubbles are round, beautifully round, a perfect shape, as in Figure 1a. This round shape is the optimal, least-energy, least-area way to enclose a given volume of air, as was proved mathematically by Schwarz in 1884. A perfect mathematical sphere, as rendered in Figure 1b by John M. Sullivan, enhanced by simulated lighting, makes for the perfect soap bubble.

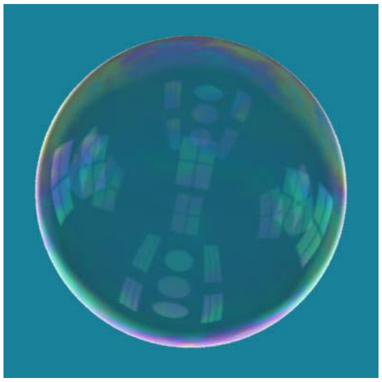


Figure 1: *Soap bubbles are beautifully round.*
a. *4freephotos.com*; b. *John M. Sullivan, used by permission, all rights reserved*

When two soap bubbles come together, they form the familiar double bubble shape of Figure 2.

Figure 2: *The double bubble. sxc.hu*

Question: is this standard double bubble the optimal, least-area way to enclose and separate two given volumes of air? A 1990 undergraduate thesis by Joel Foisy stated this conjecture.

Double Bubble Conjecture: *The standard double bubble is the least-area way to enclose and separate two given volumes of air.*

On the other hand, might something completely different do better? What are some other possibilities? Two separate bubbles as in Figure 3 are less efficient, because when they come together they can share the common wall. A bubble inside a bubble is even worse: if you move the inner bubble out, the outer bubble gets smaller.

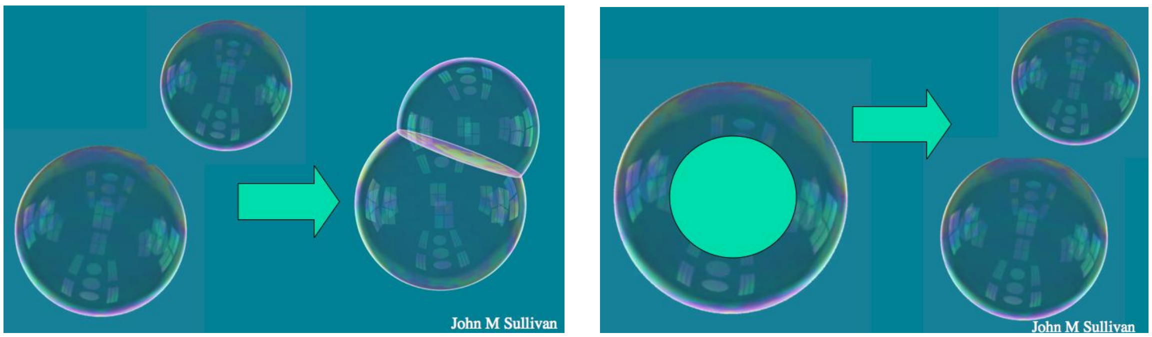


Figure 3: *Two separate bubble or worse a bubble inside a bubble is inefficient.*

Are there any other possibilities? Yes, but none that we've ever seen. To describe them, we cannot rely on photographs. Figure 4 shows an exotic double bubble, with one bubble on the inside, with a second bubble wrapped around it in a toroidal innertube. Now this double bubble is unstable and has much more area than the standard double bubble. So it doesn't contradict the conjecture. But it does make you realize that there may be many other possibilities which neither we nor the bubbles have thought of yet.

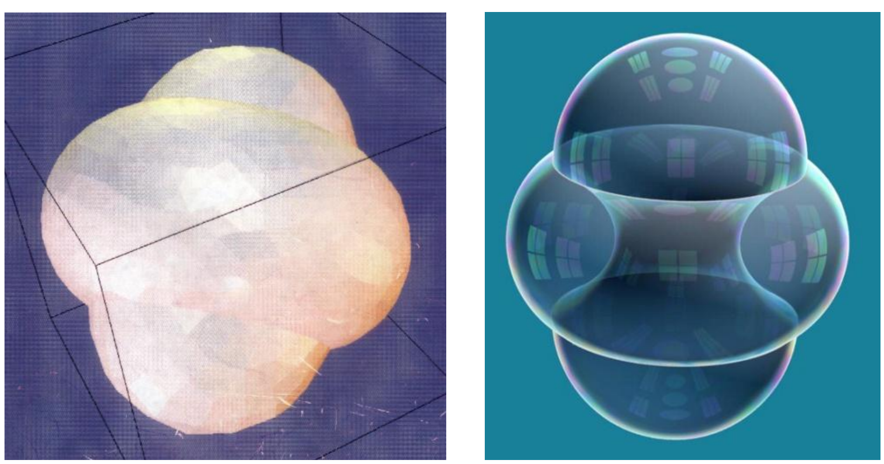


Figure 4: *An exotic double bubble with one bubble wrapped around another.*
John M. Sullivan, used by permission, all rights reserved.

There are more possibilities. Maybe as in Figure 5 the first, blue, inner bubble could have another component, a thinner innertube wrapping around the fatter red innertube, connected to the inner bubble by a thread of zero area, if you like. Or maybe there could be layers of innertubes on innertubes. Or maybe the bubbles could be knotted as in Figure 6.

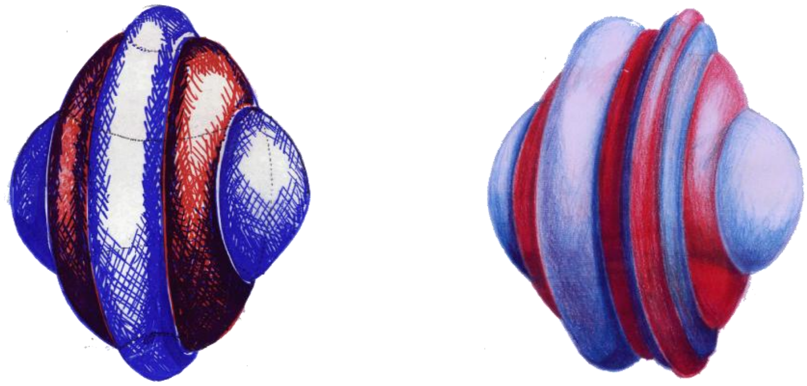


Figure 5: *Layers of innertubes.*
Drawings by Yvonne Lai, former undergraduate research student, all rights reserved.

Figure 6: *Bubbles knotted about each other.*

Or maybe as in Figure 7 the double bubble could be totally fragmented into millions of pieces, maybe with empty space trapped inside.

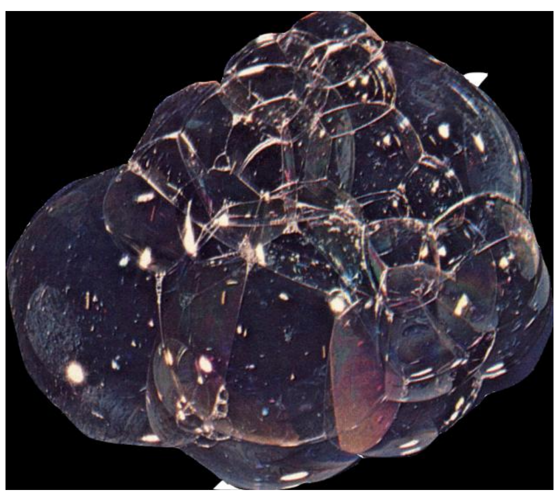


Figure 7: *A fragmented double bubble.*
Photo by F. Goro, used by permission, all rights reserved.

Alas there are innumerable possibilities to rule out in order to prove that the conjectured standard double bubble is best. Yet in this gallery of possibilities there shines a ray of hope: they all look unstable and very area expensive. On this basis the work to narrow down the possibilities went forward. First of all, a proof outlined by Stanford mathematician Brian White showed that the minimizer has to have lots of symmetry, has to be a surface of revolution. Starting from this proof, Michael Hutchings, a former undergraduate research student, now Professor of Mathematics at the University of California at Berkeley, showed that the total number of components is at most three, as in Figure 5a, although they could, in principle, be quite lopsided. The final argument, developed with my collaborators from Granada, Spain, Manuel Ritoré and Antonio Ros, proved the cases of one or two innertubes around a central bubble unstable and therefore not minimizing.

The instability proof, which we'll describe in the case of one innertube, is suggested in the working illustration of Figure 8, actually of rather high quality among the kind of scratchwork used by mathematicians. The bubble on the left has a yellow innertube about it from top to bottom. The way to reduce area and thus prove instability is to rotate the left half to the left and the right half to the right. The top gets fatter, the bottom gets thinner, but the net volume of each bubble remains the same. At the joints at top and bottom, cusps form, which can be smoothed to reduce area slightly. For more information, see [1].

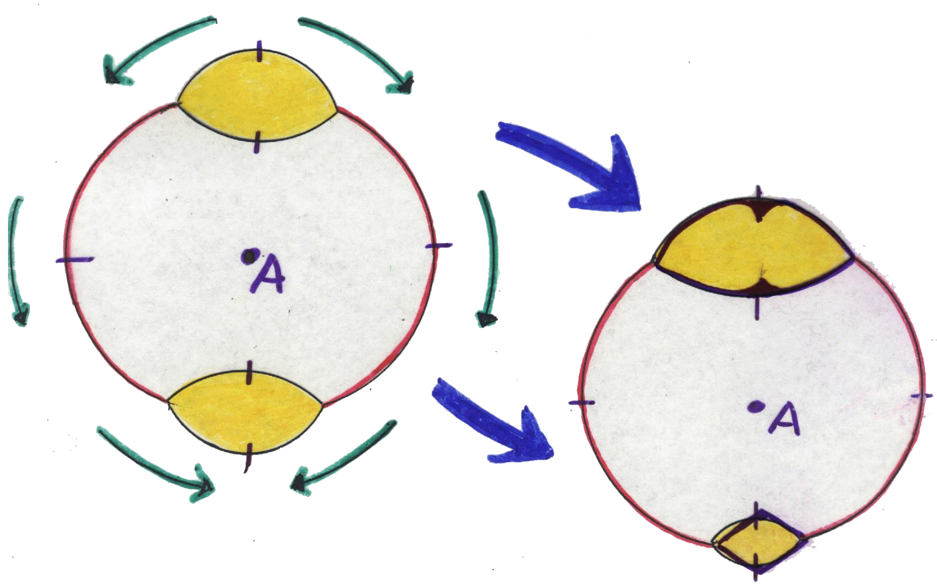
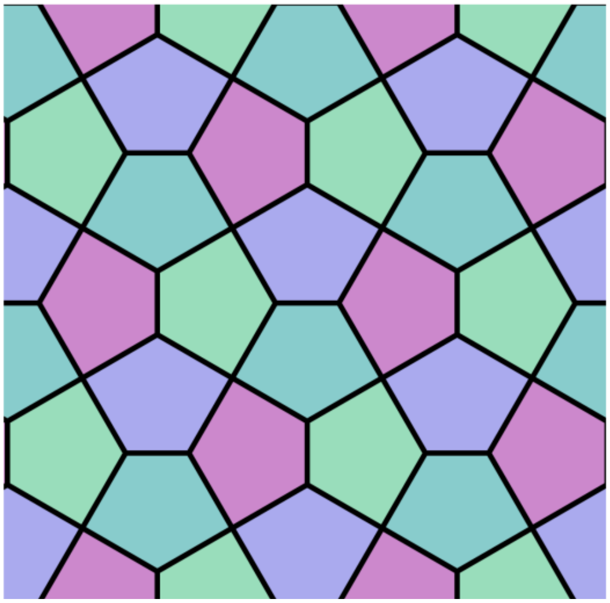


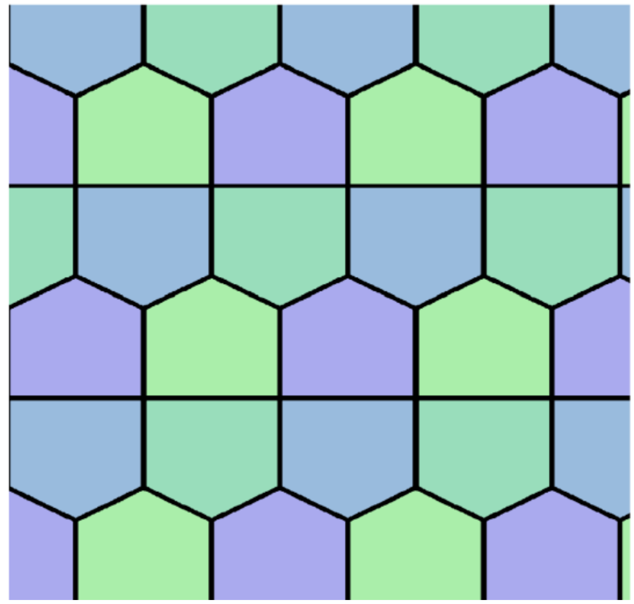
Figure 8: *This exotic bubble can be shown to be unstable by rotating the left half to the left and the right half to the right.*

Tilings

Tilings of the plane have intrigued mathematicians and architects for millennia. Although tilings by triangles, squares, and hexagons are the most common, tilings by pentagons are especially interesting and beautiful. Figure 9 shows my two favorite tilings by pentagons, the Cairo tiling and the Prismatic tiling.



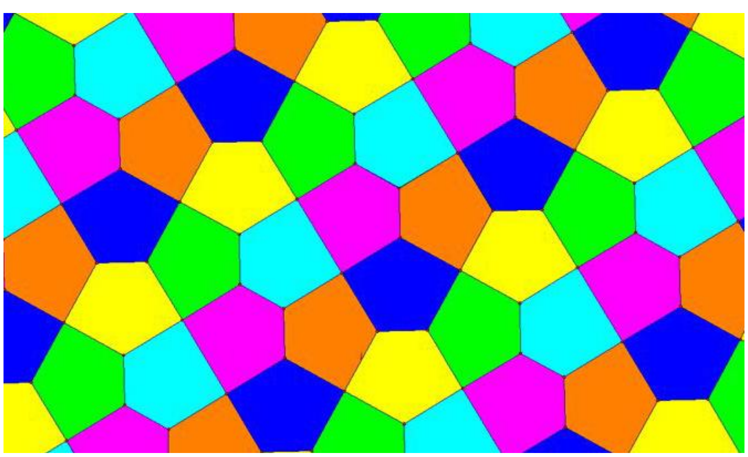
Cairo Pentagonal Tilling



Prismatic Pentagonal Tiling

Figure 9: *Efficient pentagonal tilings.*

Both the Cairo and the Prismatic tile have two 90-degree angles and three 120-degree angles. The 90-degree angles are adjacent in the Prismatic tile but not in the Cairo tile. These two tilings are in some sense mathematically perfect. Among tilings by unit-area convex pentagonal tiles, they minimize perimeter or the amount of grout required between them, as I proved in collaboration with eight undergraduate students [2]. I then issued a challenge to prove further that you couldn't tile the plane with a mixture of these two tiles. In short order, an undergraduate at MIT, Brian Chung, proved me wrong by finding an infinite family of such mixtures, consisting of alternating diagonals of Cairo and Prismatic tiles, as in Figure 10. The Cairo tiles are grouped in hexagons of four, while the Prismatic tiles are grouped in twos. Uncountably many other such tilings may be obtained by alternating variable numbers of copies of diagonals of one type with variable numbers of copies of diagonals of the other type. In the second tiling, each diagonal of Cairo tiles followed by three diagonals of Prismatic tiles.



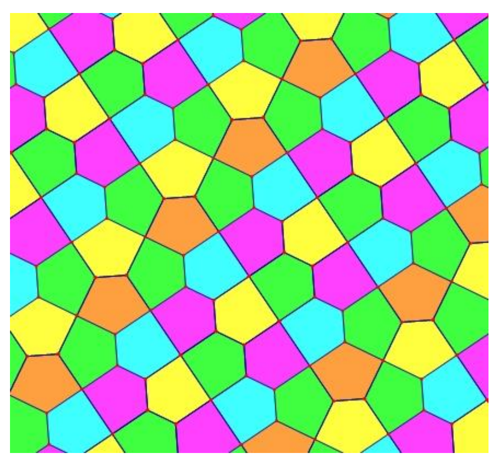


Figure 10: *Two of an infinite family of mixtures of Cairo and Prismatic tiles.*

The students believed that these were the only possible mixtures. Their method of proof was to assume there was another, start to build it, and reach a contradiction. Instead, they found another tiling, pictured in Figure 11.

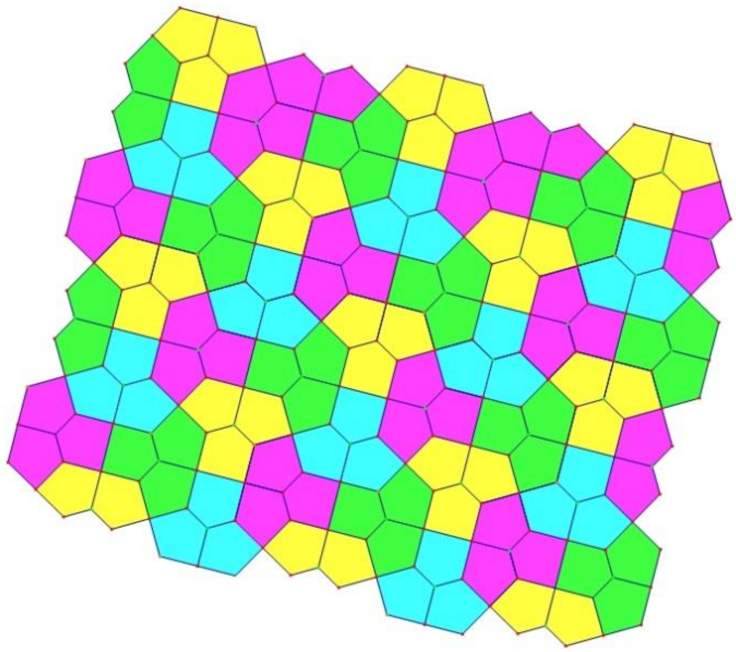


Figure 11: *Another Cairo-Prismatic tiling, "Pills."*

Then they came across the earlier example of Figure 12 on the webpage of the amateur mathematician Marjorie Rice. She found these tiles not because she was trying to minimize perimeter but just because they made such beautiful tilings.

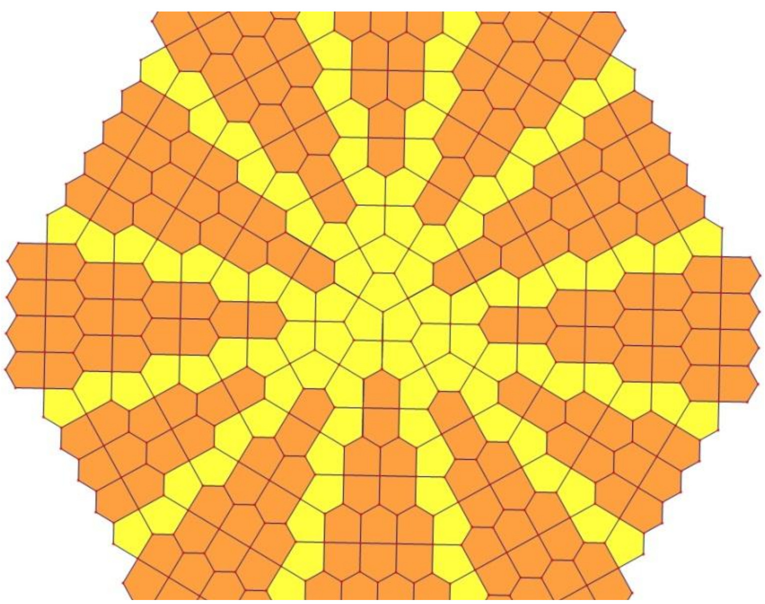


Figure 12: *An earlier Cairo-Prismatic tiling by amateur geometer Marjorie Rice.*

Eventually, by trial and error with Geometer's Sketchpad, they found many other such Cairo-Prismatic tilings with symmetries of four of the seventeen wallpaper groups and others with fewer or no symmetries. Those of Figure 13 have threefold symmetry.

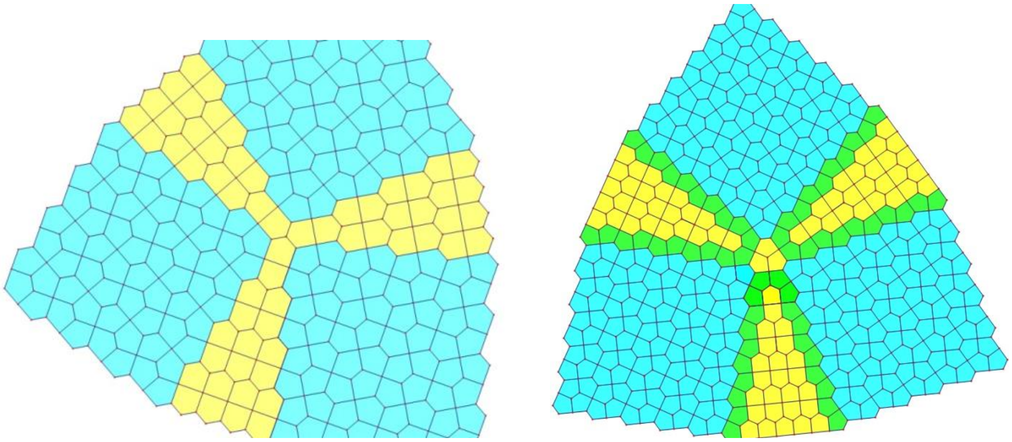


Figure 13: *"Windmill" and "Waterwheel" have three-fold symmetry.*

Others as in Figure 14 have translational and rotational symmetry. Some as in Figure 15 have only vertical and horizontal reflectional symmetry.

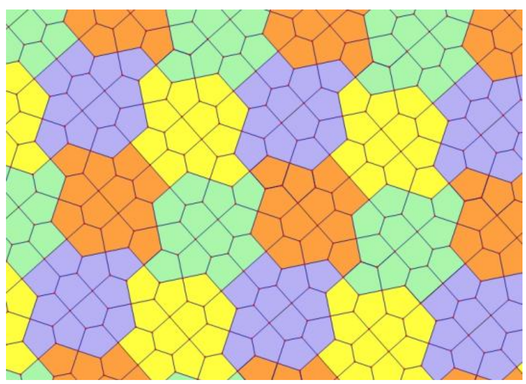


Figure 14: *"Spaceship" has translational and rotational symmetry.*

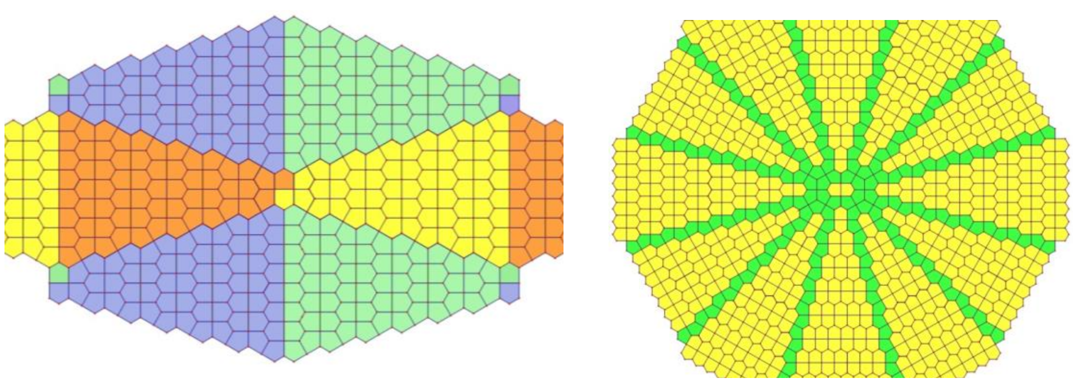


Figure 15: *"Christmas Tree" and "Plaza" have only vertical and horizontal reflectional symmetry.*

Figure 16 features "Chaos," with no symmetry at all, and the students' favorite, "Bunny," with the two bunny ears at the top.

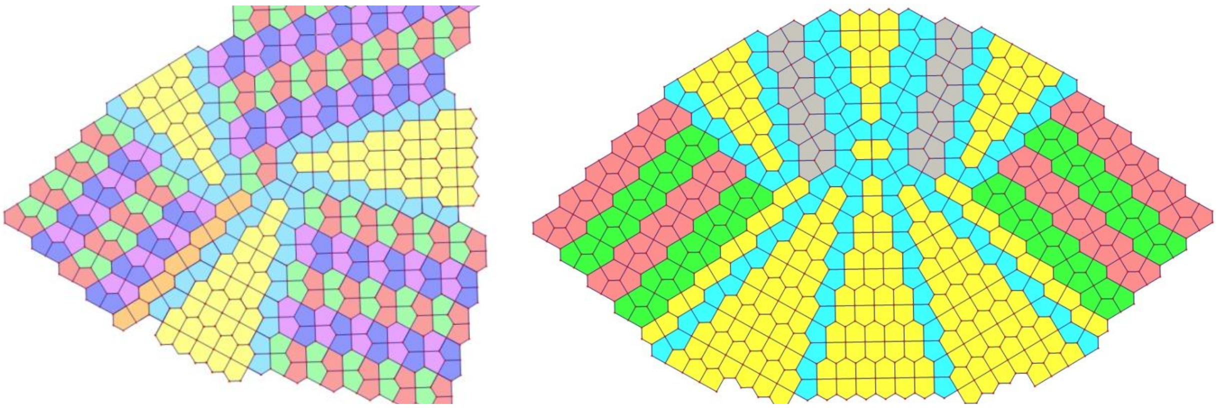


Figure 16: *"Chaos" and "Bunny."*

The discovery of these beautiful tilings required a combination of logical deduction and artistic development of the possibilities, one of the secrets of good mathematics and of good art.

References

[1] Frank Morgan, Geometric Measure Theory, Academic Press, 2009.

[2] Ping Ngai Chung, Miguel A. Fernandez, Yifei Li, Michael Mara, Frank Morgan, Isamar Rosa Plata, Niralee Shah, Luis Sordo Vieira, Elena Wikner, Isoperimetric pentagonal tilings, Notices Amer. Math. Soc. 59 (2012), 632-640.