A Fractal Wallhanging

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Abstract

A family of fractals is generated by a process known as the Iterative Function System (IFS). In this workshop the participants will create a large scale 64 in x 64 in wallhanging of a fractal from this system. Information will also be given to determine the code that enables the other fractals from this system to be recreated at large scale

Creation of a fractal by the Iterative Function System

Fractals are objects that are self-similar at all scales. To create a fractal by the *Iterative Function System* [1], begin with an arbitrary design drawn within a 1 unit x 1 unit square and iteratively subject this design to a given set of transformations. Four fractals generated by IFS are shown in Figure 1. We show how to generate the fractal in Figure 1d [2].

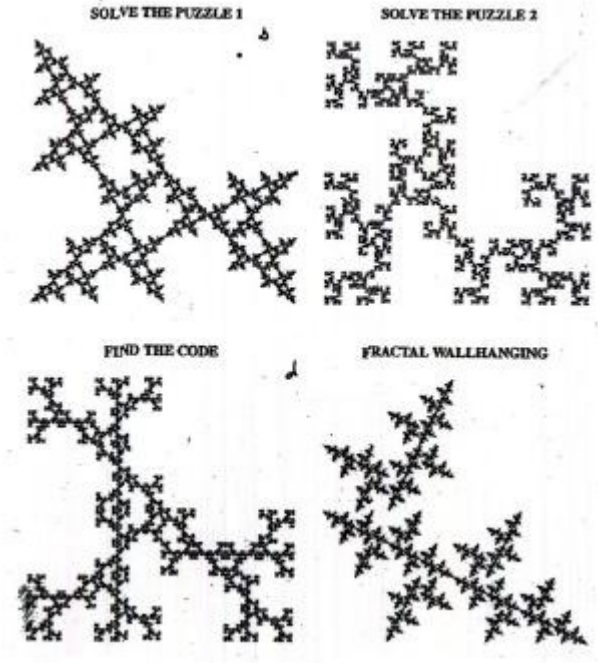

Figure 1

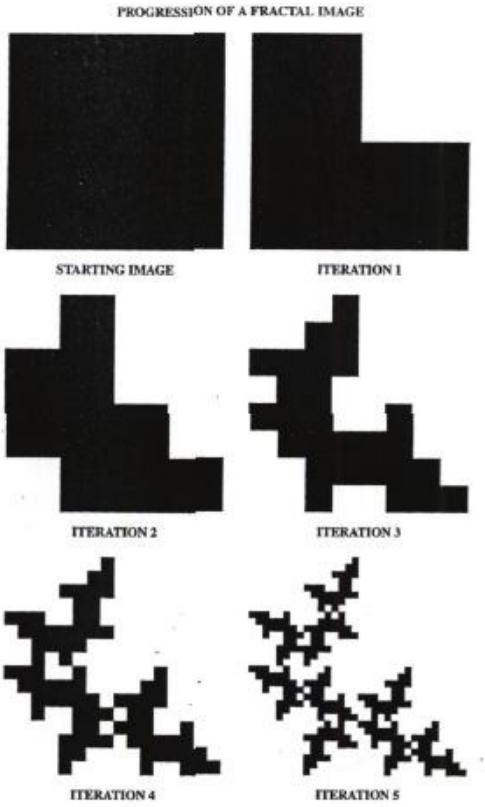

Figure 2

a. Begin with a 1 x 1 square colored black so that the starting design is as shown in Figure 2a.
b. Reduce the black square to half its size and,
   i. transform it by the identity, I, and place it in the lower right-hand quarter of another 1x1 square;
   ii. rotate it 90 deg. counterclockwise, R_{90}, and place the result in the upper left-hand quarter of the 1 x 1 square;
   iii. rotate it by a half-turn, R_{180}, and place it in the lower left-hand quarter of the 1 x 1 square.
   iv. leave the upper-right hand quarter empty as shown in Figure 2b. This results in an L-shaped pattern.
c. Reduce the L-shaped pattern within the 1 x 1 square from step b to half its size and repeat the three transformations given in step b to get the pattern in Figure 2c.
d. The next three iterations are shown in 2d,e,f.
e. If the three transformations of step b are iterated infinitely the result will be the fractal pattern shown in Figure 1d. Of course the iterations must be finitely terminated, as they are in Figure 2f after five iterations, resulting in an approximation to the fractal.

Remark 1: Each iteration reduces the size of the original square by a factor of ½. After several iterations the initial design within the square will be reduced to the size of a pixel. Since any design within the square is eventually reduced to the size of a pixel, you will end up with the same result regardless of the design that you start with.

Remark 2: If you take the final result of this process, shown in Figure 1d, notice that it is replicated in the lower-right hand quadrant. Rotate it by 90 deg. counterclockwise or by a half-turn you will notice that the pattern is identical to the patterns in the upper left-hand and lower left-hand quadrants of the original. This self-similarity is a property of all fractals.

Since this fractal was generated by the transformations (I, R_{90}, R_{180}) we will refer to this sequence of transformations as the *genetic code* of the fractal.

A Fractal Wallhanging

In the genesis of the above fractal the pattern is continuously scaled to half its size, so that miniature copies of the final fractal appear at smaller and smaller scales within the whole fractal. Now that you have approximated the fractal, you can recreate it at large scale as a wallhanging as follows:

Materials: scissors, a stack of sheets of 2 in x 2 in fractals of the type shown in Figure 1d, a stack of 8x10 sheets of paper, scotch tape, glue stick, a large 3 ft x 3ft or 5 ft x 5 ft chipboard.

a. Cut out 64 copies of 8 in x 8 in squares from the 8 x 10 sheets of paper.
b. Cut out three 2 in x 2 in copies of the fractal in Figure 1d.
   i. On the table place the first copy into the lower right-hand quarter of a 4x4 square.
   ii. Rotate the second copy 90 deg. counterclockwise and place it in the upper left-hand quarter of the 4x4 square.
   iii. Rotate the third copy by a half-turn and place it in the lower-left hand quarter of the 4x4 square.

iv. Leave the upper right-hand quarter of the 4x4 square empty.

c. Create two more copies of this 4x4 square.

d. Apply the same three transformations as in part b to the three 4 x 4 squares from step c to get a fractal pattern within an 8x8 square. You will need nine copies of the original 2x2 fractal to carry this out.

e. Attach with glue stick the nine9 copies of the fractal from step d onto one of the 8 in x 8 in square sheet of paper.

f. Repeat steps b and c two more times using a total of 27 copies of the 2x2 fractal. This step can be greatly eased if you have access to a Xerox machine

g. Apply the three transformations to three copies of the 8 x 8 square from step f resulting in a 16x16 square requiring 27 copies of the original fractal. Tape these three 8 x 8 squares together on their backside along with a blank square in the upper right-hand quadrant to form an enlargement of the original 2 in x2 in fractal to one that is now 16 in x16 in.

h. Create two additional copies of the 16 in x 16 in fractal.

i. With three copies of the fractal from steps h move up one more level and create a 32 x 32 fractal.

j. At this point you can attach this 32 x 32 square to a 3 ft x 3 ft. chipboard. But if you are ambitious and still have some energy, you can go up one final step to create a 64 x 64 fractal image.

Since the transformations to create this fractal were all rotations or the identity transformation, the transformations required to create the fractal can be carried out by movements of the original fractal in the plane. This would not have been possible if one or more of the transformations was a reflection. The creation of such fractals are much more difficult, and we will not address them in this workshop. However, we will show in the next section how to determine the transformations necessary to generate such a fractal, i.e., its genetic code.

Finding the code

The possible transformations employed by the IFS system are the eight *symmetries of a square*. These are the distance preserving transformations that keep the square invariant: counterclockwise rotations of 90 deg., 180 deg., 270 deg. : R_{90}, R_{180}, R_{270}, reflections in the vertical, horizontal, and positively and negatively leaning diagonals: V, H, D_+, D_- and the identity transformation, I, as shown in Figure 3. These transformations are known as *isometries.* Any one of the three transformations such as the ones in Section 1b can be one of these eight isometries so that there are 8 x 8 x 8 possible genetic codes. To aid in the process of finding the genetic code of an IFS fractal, use the *symmetry finder* shown in Figure 4. To use this finder, glue the fractal pattern to the left panel of the symmetry finder and its mirror image to the right panel. Cut it out and fold it in half so that the fractal is on the front and its mirror image is on the back. Each time you carry out a transformation by one of the symmetries of a square, record that isometry on the tab at the bottom of the finder as shown in Figure 4. For example the pattern on the left panel is labeled I while its mirror image is labeled V. This has been done for the fractal in Figure 1c . The fractal and its mirror image are attached to the finder in Figure 5a. Rotate and reflect the finder until you see the pattern in each quadrant of the original fractal emerge. The transformation will be listed on the tab at the bottom of the finder. Check to see that the genetic code for this fractal is (D_-, R_{180}, V). Now one can reconstruct a scale model in three of the quadrants of a square as shown in Figure 5b by subjecting the fractal to the transformations from the genetic code.

Problem: Find the genetic code for the fractal in Figure 1b using the symmetry finder in Figure 6.

Conclusion

In this workshop we have seen both how to generate fractals by the Iterative Function System beginning with a single black square, enlarge the fractal to create a wallhanging, and find the genetic code of a fractal using the symmetry finder.

References

[1] M. Frantz and A. Crannell, *Viewpoints*: *Mathematical Perspective and Fractal Geometry in Art* . 2011.

[2] V.G. Fegers ad .B. Johnson, *Fractals-Energizing the Mathematics Classroom* , in *Fractals, Graphics, and Mathematics Education* by M.L.Frame and B.B. Mandelbrot, 12, pp. 19-25. 1979.

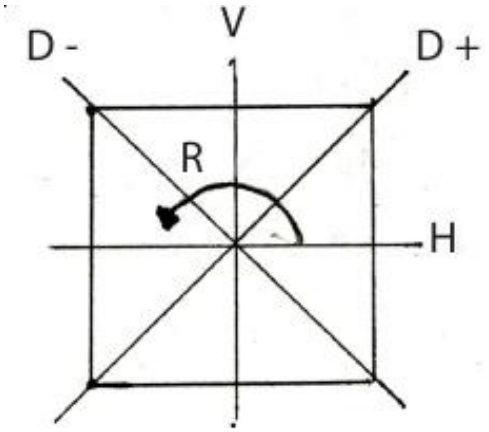

Figure 3

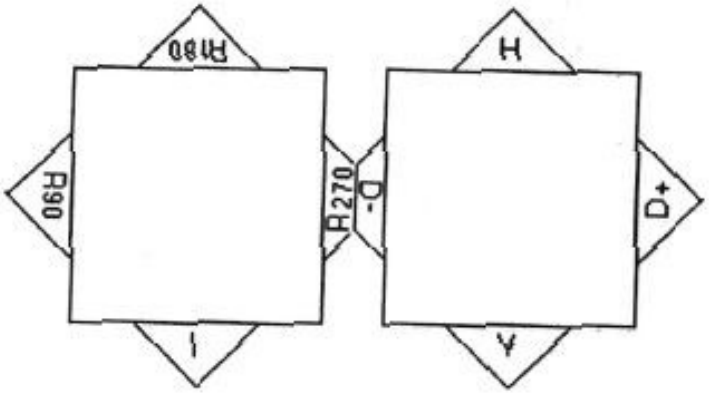

Figure 4

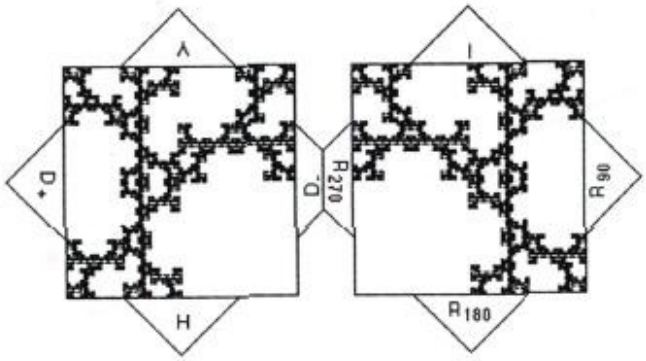

Figure 5 (a)

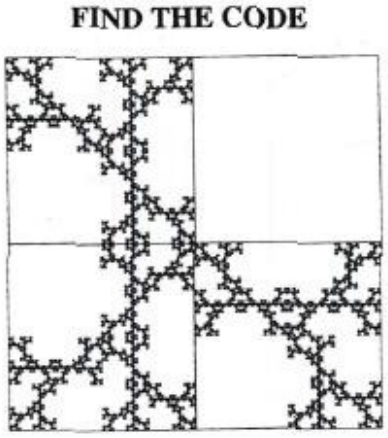

Figure 5 (b)