Flipbook Polyhedra

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Abstract

We present a flipbook exercise that helps students visualize relationships between polyhedra that share symmetry groups. Students draw progressive truncations of regular polyhedra, and these drawings are combined into animated flipbooks of the transformations. Extensions of our exercise can demonstrate other geometric operations, such as stellation and dualization.

Introduction

In this paper, we describe a two-dimensional flipbook exercise helps students visualize the relationships between polygons and polyhedra. The exercise encourages intuitive understanding of polyhedral geometry and illustrates how polyhedra within the same symmetry group relate to each other.

After completing our proposed workshop, students understand the truncation operation, and, to a lesser extent, the geometric dual. In particular, they learn that regular polygons truncate to themselves and are self-dual, and that the cube and octahedron are closely related and share rotational and reflectional symmetries.

Our paper is organized as follows. First, we survey the related literature. Then, we present our workshop exercise, and explain its mathematical significance. Finally, we describe a variety of additional flipbook exercises and extensions.

Related Work

Flipbook-style visualization, i.e. sequencing of gradually varying images, is commonly used to illustrate concepts in applied mathematics. Static "filmstrip" flipbook-style image sequences have been used, for example, in visualizing solitons [10], dynamic trees [1], and social networks [8]. Computer-animated flipbook-style visualizations have appeared as well, depicting concepts such as surface morphing [9] and object-oriented program execution [2]. Flipbook-style visualization tools have also been used in education (e.g., for teaching chemistry [4]). Pursuant to these applications, a number of software tools have been delveloped to produce flipbook-style visualizations (e.g., the Palais [9] `3D-Filmstrip` system).

Animated flipbooks themselves have a long history—they were first patented in 1868 under British Patent No. 925—and have at times been used for mathematics instruction. Cernek and Williams [3], for example, present a flipbook exercise in which students practice artihmetic by drawing a flipbook in which successive pages have increasingly many objects. To our knowledge, however, our work is the first to introduce animated flipbooks as a tool for teaching geometric concepts.

Our geometric flipbook exercise allows students to engage with each stage of the truncation construction. As Hanson *et al.* [5], Hundhausen *et al.* [7], and Badame and Boothe [1] have pointed out, such interactive visualizations at improve student understanding far more effectively than static visulizations do.

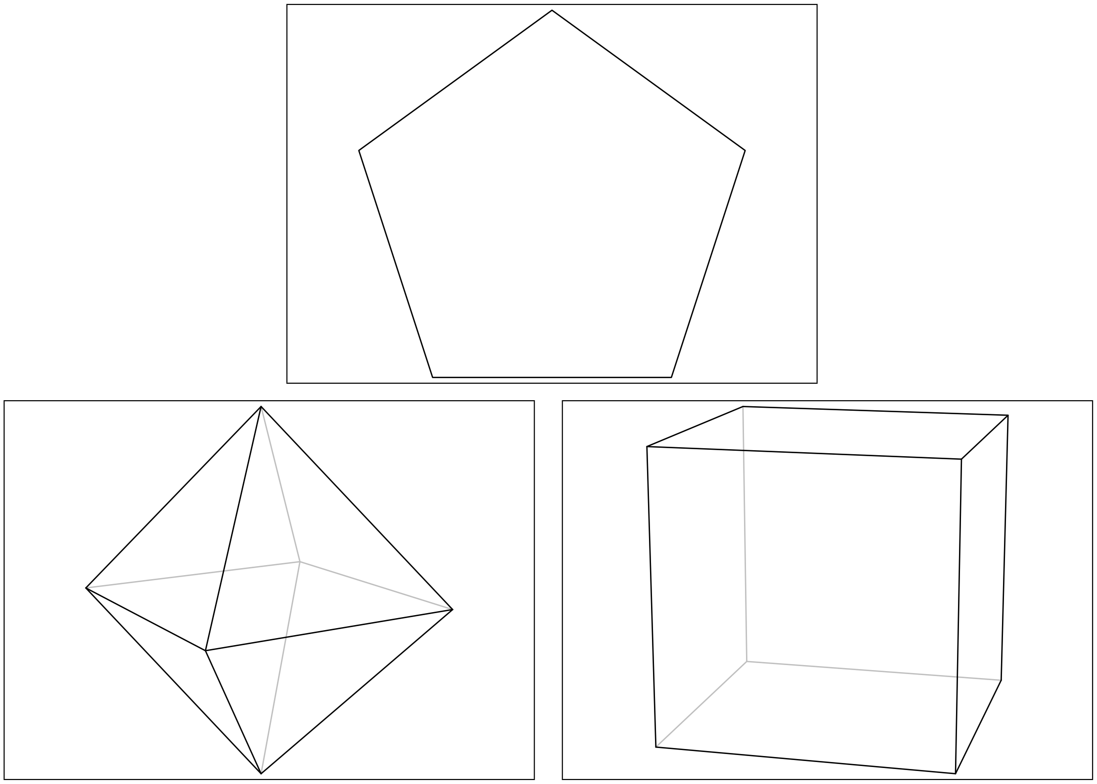


Figure 1 : *Three templates to be used for flipbooks' final pages.*

Materials

Each student should either make or be provided bound flipbooks prepared according to the instructions below. Students will also need a pencil, a pen, and an eraser. They may also use markers or colored pencils, for coloring in their flipbooks. Straight edges may be helpful, but are not strictly necessary.

Preparation of Flipbooks. Flipbooks are 5cm by 7cm large. They are composed of eleven sheets of blank white paper, followed by a template sheet that is used to initiate the exercise. Example template sheets are presented in Figure 1.

Flipbooks should be stapled together along one edge. To aid in flipping, it is helpful, but not strictly necessary, to make a bias cut along the edge opposite the binding. To make a bias-cut, bend the flipbook into a U-shape, as shown in Figure 2a. Then, make a straight cut as shown in Figure 2b. This will result in a bias-cut flipping edge as pictured in Figure 2c. Note that a bias-cut flipping edge will flip more easily than a straight-cut flipping edge, but will flip in only one direction.

Workshop Description

We describe an example flipbook exercise using the template sheets provided in Figure 1: a regular pentagon, octahedron, and cube. Students create truncation flipbooks using these templates by progressively truncating the template figures.

For simplicity, we will refer to pages in the flipbooks by number, with page 12 (the last page in the flipbook) being the template page. The truncation should span pages 2–12; flipbooks' front pages may be used for title and byline.

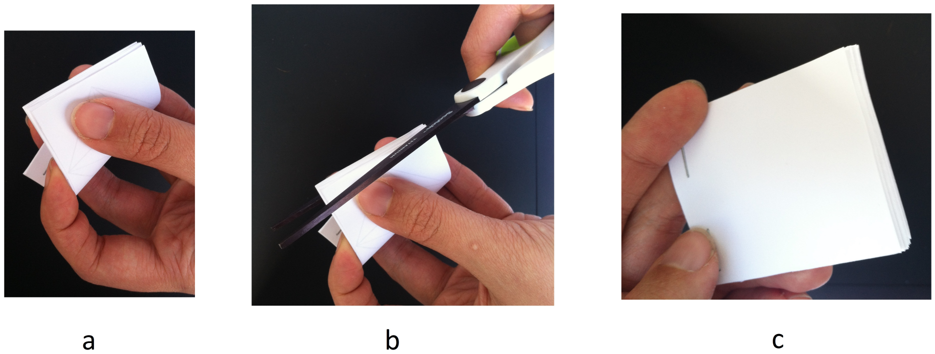


Figure 2 : *Method for creating a bias-cut flipping edge.*

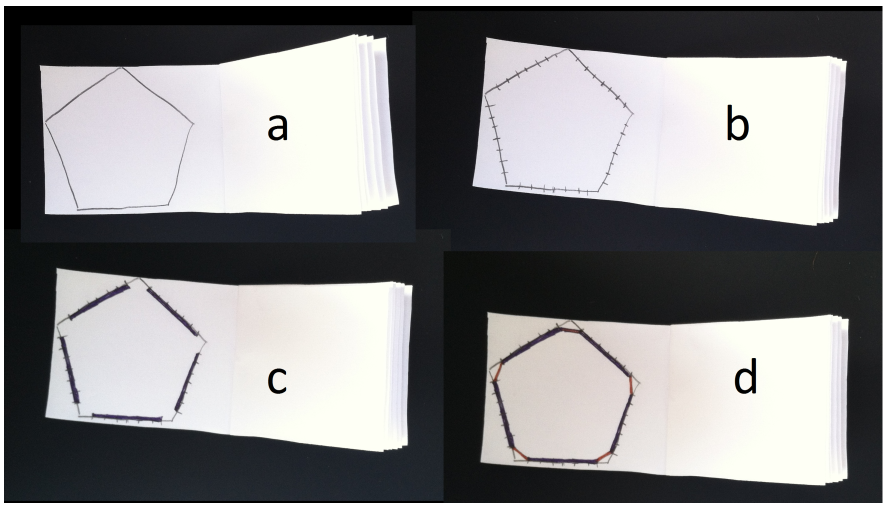


Figure 3 : *Images of a pentagon truncation.*

Students begin by creating pentagon flipbooks that introduce the technique. The procedure is as follows.

1. Beginning with page 11 and working forwards, students trace the pentagon template onto each page with a pencil. (This tracing is erased later.)
2. Students then divide each side of each traced pentagon into eighths. This can be done carefully using a ruler, or by successively eyeballing midpoints.
3. At this point, the students switch to using pen. They ink in two copies of each step of the truncation, starting from the back of the flipbook. On pages 10 and 11, the full pentagon should be inked.
4. On pages 8 and 9, students should ink the sides of the pentagon only between the $\frac{1}{8}$ and $\frac{7}{8}$ marks. They should then connect the line segments to each other to form a decagon (or partially truncated pentagon), as seen in Figure 3.
5. Pages 6 and 7 are completed similarly, except with the pentagon inked only between the $\frac{1}{4}$ and $\frac{3}{4}$ marks.
6. Pages 4 and 5 are inked only between the $\frac{3}{8}$ and $\frac{5}{8}$ marks.
7. On pages 2 and 3 (the full pentagon truncation), the midpoints of each side of the traced pentagon should be joined together to form a smaller regular pentagon.
8. At this point, the pentagon flipbook is completed, and students may erase the traced pentagons. They may also and add decoration and color.

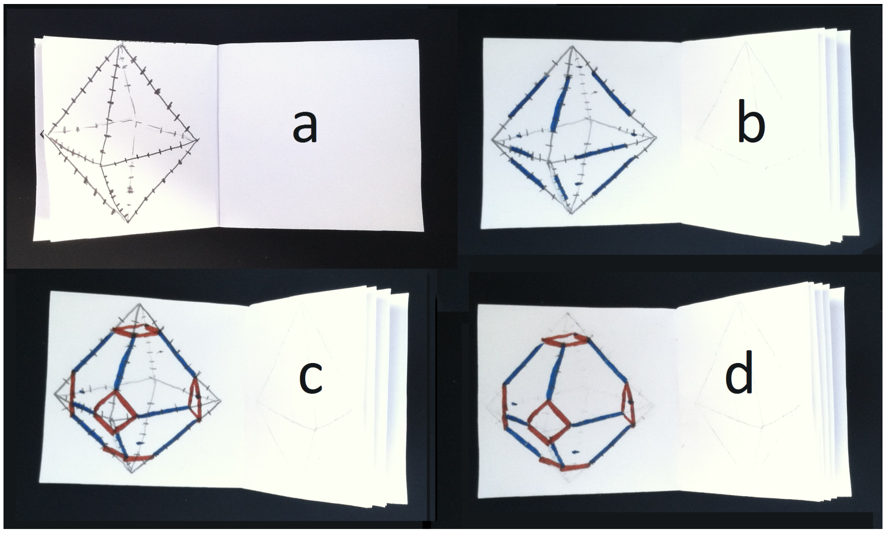
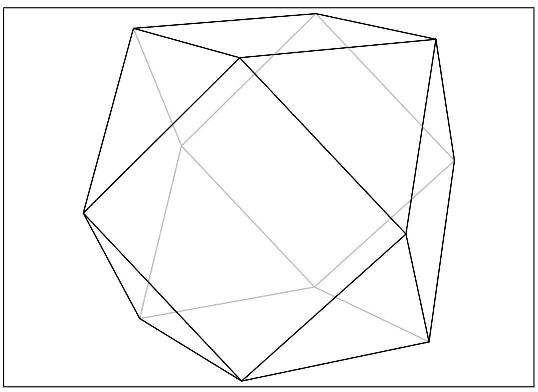


Figure 4 : *Images of an octahedron truncation.*



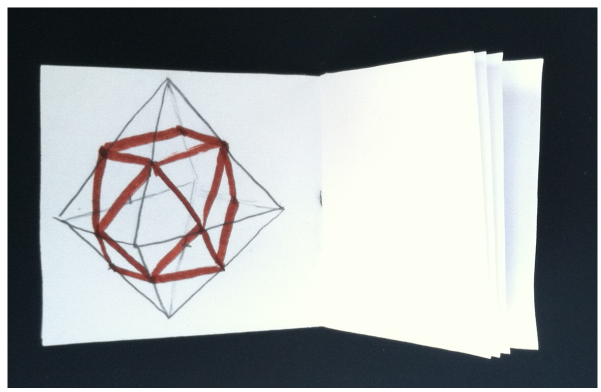


Figure 5 : *The cuboctahedron resulting from truncating both the octahedron and cube pictured in Figure 1.*

Once students have completed the pentagon exercise, they should move on to the more advanced exercise of truncating the octahedron and cube. Because flipbooks are necessarily two-dimensional, students will now have to account for perspective and occluded faces in their drawings.

The octahedron and cube truncations begin in the same manner as the pentagon: Students trace the template figure onto all the pages of the flipbook. Students then continue by dividing the sides of the traced figures into eighths. Note that because of the projection, the polyhedral figures' sides are not of the same length, and the lengths of the subdivided eighths are thus also not equivalent.

The inking process proceeds similarly to the polygonal case, except that now students must account for the truncation of occluded faces. In particular, pairs of pages are inked along the penciled guidelines between progressively smaller sets of markings, and the resulting segments are joined together. Students may find it helpful to pencil in a wireframe projection to aid in accounting for the truncated faces.

An example octahedron truncation is pictured in Figure 4.

After finishing the octahedron and cube flipbooks, students may be surprised to discover that they end up with the same figure at the ends of each: the cuboctahedron pictured in Figure 5.

Mathematical Significance

Duals and Truncations in Two Dimensions. Our first flipbook exercise shows that the truncation of a regular pentagon is another regular pentagon. As the full-truncation of a pentagon joins the midpoints of the sides, this truncation is equivalent to taking the dual of the original pentagon. Thus, the pentagon flipbooks illustrate that the regular pentagon is self-dual. Thinking carefully about the last step of the process—connection of the sides' midpoints—students should realize that an analogous relationship holds more generally: the full-truncation (and dual) of an n-gon is an n-gon, and all regular polygons are self-dual. Similarly, considering the truncation process, they should realize that the partial truncation of an n-gon is always a $2n$-gon.

Symmetry. Performing truncation operations on platonic solids can help students gain intuition for major symmetry groups in three dimensions. The truncation operation is symmetry-preserving. Students can gain intuition for this fact by observing, while inking in truncated figures, that truncation occurs equally at all corners. Thus, all the figures within a given truncation flipbook have the same rotational and reflective symmetries and share a symmetry group.

The octahedron and cube for which we provided templates were specifically chosen to truncate to the same view of the cuboctahedron (Figure 5). Students should thus realize through the flipbook exercise that the cube and the octahedron share a symmetry group. You may wish to give more detail about the octahedral symmetry group (O_h) to advanced classes.

If students spend more time considering what the cube and octahedron have in common, they may realize that the octahedron can be lined up such that each vertex aligns with the midpoint of a face of the cube. Students may recognize that this alignment is remniscent to the operation of joining the midpoints of sides in two dimensions to obtain the dual; they may then intuit that the cube and octahedron are mutually dual.

Extensions and Conclusion

The templates provided give just a few of the many possible flipbooks that can be generated with just the truncation operation. Truncation flipbooks of other regular polygons and polyhedra can be drawn, and students may find it interesting explore the truncatations their favorite figures. With good alignment, it is possible to merge a cube-to-cuboctahedron flipbook with an octahedron-to-cuboctahedron flipbook to generate a cube-to-cubeoctahedron-to-octahedron flipbook that exhibits both truncation and expansion.

Alternative geometric operations can be demonstrated through flipbooks, as well: Stellation flipbooks can be constructed through successive extension of the template figure's edges. Dualizing can be demonstrated through flipbooks as well. Students might experiment with dualizing irregular n-gons to find, for example, that the dual of a rectangle is a rhombus, and, more generally, that the dual of a polygon with regular angles is one with regular sides.

Flipbook construction can be combined with a deeper conversation about the derived geometric animations.[1] Students can be asked imagine the truncation process before constructing their flipbooks. Then, they can compare the true operation, as represented in the flipbook animation, to their imagined visualizations.

More generally, flipbook construction is a novel interactive teaching exercise that demonstrates how doodle games ([6]) can help students develop mathematical intuition. It may be useful for illustrating a variety of geometric concepts beyond properties of polygons and polyhedra.

[1] We thank the referees for this suggestion.

Acknowledgements

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