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Printing by Rolling Möbius Band Stencils: Glide Reflection Embodied in Physical Action

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Abstract

This hands-on workshop involves making ordinary untwisted band stencils and Möbius band stencils. Then it explores printing patterns with them by rolling them out. The two types of pattern are compared for symmetry and isometric transformation properties of translation and glide reflection. The materials and activities have been selected for low cost and ease of use to enable the activity's transfer to a range of classroom settings in art and or mathematics. Interlocking tiles extend these frieze patterns to ones with rotations and to rosette patterns.

1 Introduction

Escher's tessellations in the plane are well known examples of patterns made with transformations. In particular they tile the plane, with no gaps or overlaps, with systematically repeated transformations of the tile to cover the whole plane. See [3] for a mathematical approach to tiling patterns in general. Escher produced patterns with the glide reflection, and used color to demarcate different positions of the tile as it is transformed over the plane. See [8], p 140, for an illustration of such a pattern (system V^C). The sculpture 'Walk on LA' [2] makes an art work of both a roller and the repeated pattern rolled out by it imprinted in the sand on Santa Monica beach, Los Angeles. Many examples of outdoor sculptures of Möbius bands and patterns for Möbius band knitting can be found online. Issues involved in constructing Möbius bands in different media and crafts have also been described such as in [4].

In this workshop we see how the continued rolling out of an ordinary untwisted band produces repeated translations and that the continued rolling out of a Möbius band produces repeated glide reflections [1,9]. This correspondence is shown by using the ordinary untwisted and Möbius bands as stencils which are used for printing out patterns on long strips of paper. Once the pattern is established through rolling and printing the Möbius band, then the stencil is unwrapped flat and the glide reflections can be continued via combinations of translation and turning over of the flat stencil. This allows continued printing of the pattern using the flat stencil, thus highlighting the glide reflection. These repeating strip patterns are frieze patterns [3]. They can be classified according to transformations, such as translation or glide reflection, that generate them. Interlocking tiles are provided that generate these and frieze patterns generated with 180 degree rotations and also a rosette pattern [1,3].

This physical correspondence gives a real world experience and conceptualization of the generation of glide reflections. This type of physical correspondence, or embodiment, already exists for rotations (turning in the plane about a point), reflections (folding, turning over or looking in a mirror) and translations (respectively). These actions of the human body can facilitate the conceptualization and embodied cognition [5,6] of the geometric transformations. Möbius band printing gives a physical action that can facilitate the conceptualization and embodied cognition through movement of the glide reflection. This means each of the four types of isometry [9] (rigid transformation) in the plane can be conceptualized through physical actions. Material lists and suggested previous activities are covered in section 2, while section 3 gives the activities with discussion points for classroom application. To extend the activities we refer the reader to [1]. Section 4 contains additional mathematical discussion of the path traced by a Möbius band discussed in section 3.2.

Mathematically, the workshop examines three aspects and their inter-relationships: (i) the surface being rolled out, (ii) the repeated pattern created and its symmetries, and (iii) the planar rigid isometry that will transform one part of the pattern (where each part the whole surface is printed once and only once) to its adjacent position. These relationships, between surfaces and the patterns produced when they are rolled out, are studied in orbifold theory [7]. This theory includes the converse question: Given any pattern with certain symmetries, which surface, or surfaces, could be rolled out to produce it? Consider a wallpaper pattern with a small motif that is repeatedly copied and translated to fill the whole wall. This would be rolled out by a torus with the motif on it. If the motif here is asymmetric and not made up just of copies of a smaller motif, then this torus will be the smallest surface that can roll out the motif over the whole wall. If however, the motif is made up of an image with, say 180 degree rotational symmetry, then a smaller surface with one half of the motif will roll out over the whole wall [1]. A pattern with glide reflections horizontally and with translations vertically would be rolled out by a Klein bottle. Computer images of these processes of rolling out a variety of patterns from a variety of surfaces are shown in [1].

2 Materials, Equipment and prior learning

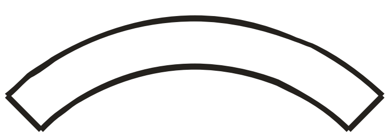
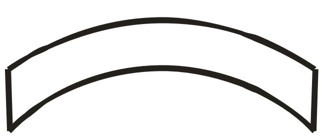
Strip for band	Path of untwisted band	Path of twisted band
		
		
		
		
		
		

Figure 1: *Table for 'rolling out' activity (3.1)*

One of the goals of the workshop is to give an experience of the embodiment of the concept of glide reflection in a physical action that produces a pattern. This action involves sliding (translation) and turning over (reflection). The turning over as an embodiment of reflection by itself can be experienced through folding paper with wet ink or paint on it to make a pattern with reflective symmetry. This reflection can be shown as a mirror image if a mirror is placed on the fold line. This could be useful for students as a lead-in to the activities in this paper. A materials list is in the appendix.

3 Workshop Activities

Part 3.1 is an exercise to explore the path a band takes as it rolls to show the effects of adding or not adding a twist. Also what if the ends of the strip it is made from are connected at an angle instead of in-line? What if the strip is curved? This is a preliminary exercise to become accustomed to the materials and to familiarize students with the properties and behaviour of bands before using them to make basic prints and examine their properties. These printing activities can be extended to rosette and wallpaper patterns as described in [1], and in section 3.5 where self interlocking tiles are used instead of a band.

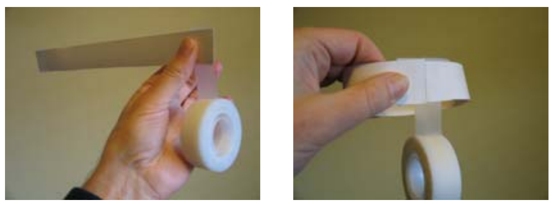


Figure 2: *Taping strips into bands*

3.1 Rolling out bands without printing. These instructions are for the bands shown in the left column of the table in Figure 1, first cut out each strip. Tape around each end edge separately covering both sides of each end in tape (Figure 2 left). This enables the strips to be taped into bands and un-taped repeatedly. Then make each band into a loop without a twist, so it looks like an untwisted elastic band (cylinder). Do this by placing the ends lined up, short edge to short edge, and fixing in place with tape over the taped ends (Figure 2 right). The tape must go across the whole width of the band, otherwise the band can hinge. Where the short ends of the strip are not both cut at right angles to the long edges of the band, make sure the short edges line up touching along their edge. Do not leave a gap or overlap between them when taping, even if this puts a kink in the band. Where one end edge is longer than the other, make sure the whole of the shorter end edge touches part of the longer end edge. Discuss which bands have a kink when taped up and why.

Ask students to predict what they think the path will look like as the band rolls along, repeatedly rolling over itself, e.g. straight line, wiggly line, etc. Then let them roll the band out, without sliding against the paper, neither moving nor twisting on the table as it rolls. There should always be a section of band held flat pressed on the table, unable to move or twist, while the rest of the band rolls over it, until it rolls over the next section held down. Then students sketch the path followed as a line going along the centre of the band where it touches the table, e.g. straight line or wiggly line etc. in the middle column of the corresponding row of the table. Ask students if there is anything about the shape or appearance of the bands that could be used to predict the shape or appearance of their paths.

Now students un-tape each band. Give one end a half twist and retape the band, again with end edges aligned. This converts them into Möbius bands. Then let the students predict the path followed. When each band is rolled out, students sketch the actual path in the corresponding row in the right hand column of the table. Ask students if there is anything about the shape or appearance of the bands that can be used to predict the shape or appearance of the paths?

3.2 Mathematical discussion of paths. Ask students for comments and observations and the differences between the paths of the twisted and the untwisted bands. Ask students why they think the twisted bands all seem to have paths that go on forever without looping back. This happens even if the band is curved or is taped at an angle. By contrast some of the untwisted bands do not go on forever without looping back as they are rolled out. Leave the question unanswered or just note any student responses to be revisited later. Also see sections 3.4 and 4.

Figure 3: *Shape punch printing*

3.3 Printing. Use a variety of shape punches (asymmetric and symmetric) on paper bands that are just wide enough for the punches (e.g. half width receipt roll) and at least 10 times longer than wide. Use **each punch once and only once** on a band. Roll out each band both as an ordinary untwisted band (Figure 3 left), and as a (twisted) Möbius band (Figure 3 right) on paper using the sponge to print the shape holes as it rolls (Figure 3 center, the band rolls from right to left). Roll each band so it loops over itself at least 3 or 4 times for each printed pattern, enough to see how it could repeat forever.

Discuss with students:

(i) What difference do they see in prints of the same shape in different parts of a pattern, and is this the same for both patterns, from the ordinary untwisted and twisted Möbius bands?

(ii) Does the kind of punch shape used on a strip effect the difference between the pattern from an ordinary untwisted band compared to the pattern from a Möbius band made using the same strip? There has to be an asymmetry to see the difference. See how the shapes in Figure 3 have differing symmetries. The cat has no symmetries, the tree and butterfly have one reflection each, and the flower has many symmetries, rotation and reflection. If only the flower is used then the two patterns will be hard, or impossible, to distinguish. The students can be asked if this is true or if they observed it and why it is true. This connects an image having reflective symmetry with the image remaining unchanged by a reflection in the line of symmetry. The flower shape, if suitably aligned on the band, will look the same after the reflection. So after the combination of translation and reflection, that make up a single glide reflection, this shape will look the same as if it had only been translated. This is untrue for asymmetric shapes. This is explored in 3.4.

An easy alternative to using a shape punch is to make tracing paper bands. Draw a pattern, or write a name, on an un-taped tracing paper strip in pencil on one side. Then copy it in pencil by tracing through to the reverse side of the strip. The strip can then be taped to make the bands. Trace the image repeatedly onto paper to make a pattern as a band rolls out. A further alternative is shown in Figure 4. A stencil was cut out of paper around a drawing. Then the paper was taped to a strip of thin plastic from a folder and cut around again to make a plastic stencil. This plastic stencil was taped to form a Möbius band and its edge was marked with the ink pad and make up sponge (see Figure 4).

3.4 Pattern analysis and embodiment of translation and glide reflections. In 3.3 two types of pattern were printed and discussed for each strip, both as an ordinary untwisted band and as a Möbius band. Now un-tape all the bands to return to the original strips that can lie flat. For each print, lay the flat strip on the start of the pattern (e.g. Figure 4) and now slide it along the pattern to the next part. Does it fit on the next part (e.g. Figure 5)? Does it need turning over to fit (e.g. Figure 6)?

Figure 4: *Start position*

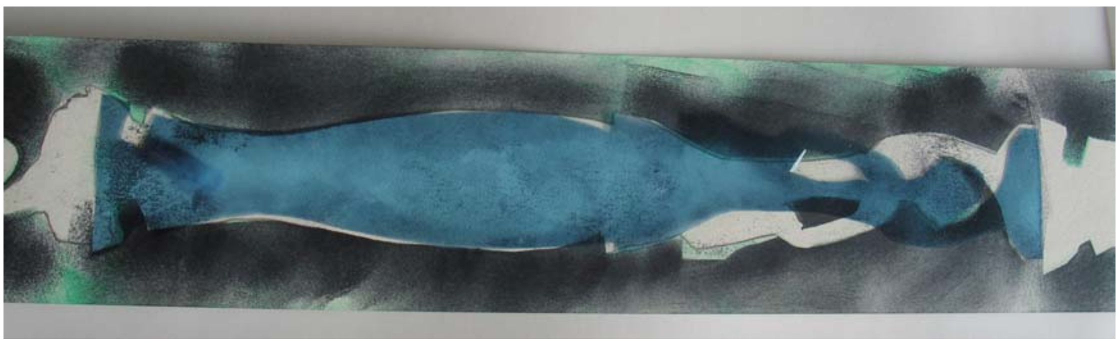


Figure 5: *After the sliding to the second printed position*

Figure 6: *After turning over to match the second printed position*

In the example of Figures 4-6, the pattern comes from a Möbius band, and it was necessary to turn the stencil over, effectively reflecting the shape it outlines after sliding the stencil to the next position. The slide (translation) combined with the turn over (giving a reflection in a line parallel to the translation) results in a glide reflection of the stencil. The fact that the reflection is in a line parallel with the translation is important and worth emphasizing in discussion. It connects to questions asked in 3.2, discussed further in 4.

3.5 Connection to tiles, isometry and frieze patterns. Figure 7 shows nine tiles. This activity investigates how copies of each tile can fit together along the interlocking edges (thin lines). There is only one way two copies of the same tile can be placed together along a shared interlocking edge, even if the tile can be turned over. If a tile is placed on paper, and its outline is drawn on the paper, there will be one unique way it has to move to an adjacent position that interlocks with one particular thin interlocking edge of its original position. Turning the tile over and moving it to its adjacent position will correspond to a glide reflection, for these interlocking tiles. Twisting it round without turning it over will correspond to rotation. For each pattern generated by each tile, discuss with students which isometry (translation, rotation, reflection, glide reflection) takes each position of the tile to its adjacent interlocking position.

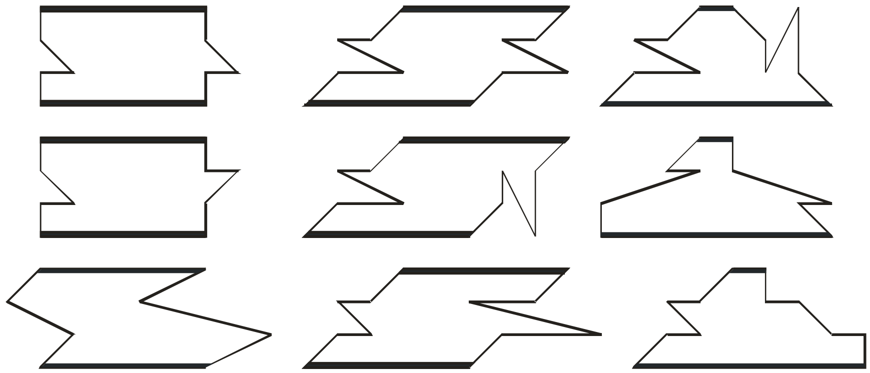


Figure 7: *Self-interlocking tiles*

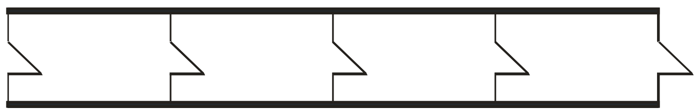


Figure 8: *Frieze pattern in an infinite strip generated from repeated translations*

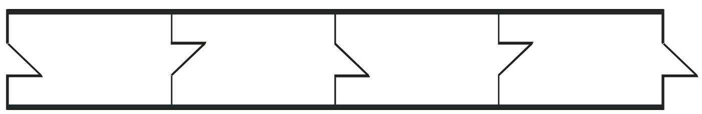


Figure 9: *Frieze pattern in an infinite strip generated from repeated glide reflections*

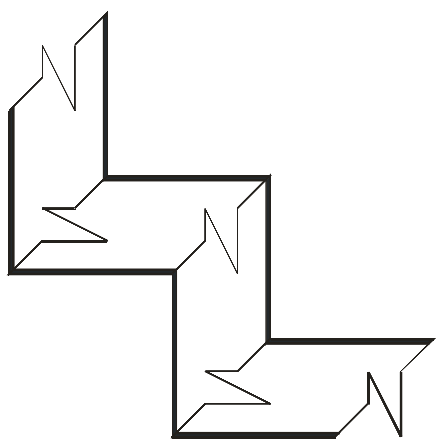


Figure 10: *Frieze pattern from repeated glide reflections in a zig-zag strip*

These kinds of repeating patterns along a strip are called frieze patterns (Figures 8-10). Figure 11 shows a tile that makes a rosette pattern [1]. It illustrates a pattern looping back on itself. The geometric transformations between adjacent positions of the tiles are rigid. This means they do not shrink, expand or distort the tile. Rigid transformations are called isometries (see [9]).

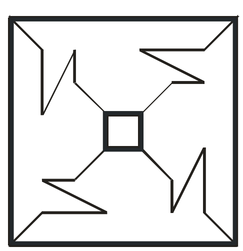


Figure 11: *Tiles that form a rosette pattern from repeated rotation*

4 Paths of the Möbius Bands

We can consider why the path of a rolling Möbius band follows a path that continues out to an infinite distance without looping back. Here is an informal argument. Consider we have cut a Möbius band so it is just a strip, or un-tape a band back to the strip it was made from. If we lay the strip out on one side, we get part of the path the band follows as it rolls out. We have observed, but not formally proven, that we can continue the path by glide reflecting the strip and laying it flat. This glide involves a reflection which means that each part of the strip that turned to left will now turn to the right, and vice versa. So the net result after both sides of the band have rolled out the same number times is for the path to be headed in the same direction as it started. That is, for every turn to the left, there has been an equal and opposite turn to the right, and vice-versa. This relates to the fact that a glide reflection done twice results in just a new, double length, translation. The two reflections cancel, and there is not net rotation or direction change.

Suppose we put a little sticker at a point $\boldsymbol{x}$ on one side of the band away from any kinks or corners. We can also mark an arrow at $\boldsymbol{x}$ pointing along the band in the direction the band will roll when $\boldsymbol{x}$ touches the table face down. We can then see where it touches the table face down one time at point on the table which we call $\boldsymbol{x_1}$, and which in direction on the table it is rolling at $\boldsymbol{x_1}$ (which way the arrow points as it is face down on the table). Then we see where it is the next time face down at point $\boldsymbol{x_2}$ on the table, and again which way to is rolling at $\boldsymbol{x_2}$. When the band rolls from point $\boldsymbol{x_1}$ to point $\boldsymbol{x_2}$ it will roll over the whole of the band on both sides. This is the same as placing the strip un-taped first on one side and then on the other. From the last paragraph we see that the band is rolling in the same direction (the direction of the face down little arrow) at point $\boldsymbol{x_2}$ as it is at point $\boldsymbol{x_1}$. This means the point $\boldsymbol{x_3}$, the third place the sticker touches the table face down, will be in line with $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, and the distance from $\boldsymbol{x_2}$ to $\boldsymbol{x_3}$ will be the same as the distance from $\boldsymbol{x_1}$ to $\boldsymbol{x_2}$. This pattern will continue for each time the sticker touches the table at regular intervals along the same straight line, hence moving off into the distance, not moving around in a circle.

For simplicity, we can exclude the possibility that the distance from $\boldsymbol{x_1}$ to $\boldsymbol{x_2}$ is zero. This possibility would arise if the Möbius band were an annulus that had been cut and re-glued with a twist.

5 Conclusion

The transformations of printing with a stencil motif on a Möbius band as it is rolled out are glide reflections at a rate of one glide reflection for each side of the band rolled. The associated physical movement of an un-taped strip stencil with an asymmetric image will involve a slide and a turning over that embody a translation and reflection that make up a glide reflection. Thus the glide reflection is embodied in physical action by a person.

The repeated application of a glide reflection can be related to the fact that the path of a Möbius band as it is repeatedly rolled out will continue out to infinite distance (Figures 9 and 10). A glide reflection done twice is a translation. A straight ordinary untwisted band when rolled out will follow a straight line path. This relates to the repeated application of a translation (Figure 8). If an untwisted band is not straight (e.g. made from a curved strip or a strip taped at an angle) then it may follow a path that loops back on itself when unrolled. This case relates to repeated application of a rotation (Figure 11).

The artistic use of these rolling out patterns hopefully adds to the experience, memory and understanding of the activity and its mathematical properties. Extensions to these activities can be based on further examples in [1].

Acknowledgments

Thanks to Eva Knoll and John Sharp for comments and suggestions.

Appendix

Material list with respective activities and subsections:

- One copy per student/group of Figure 1 for filling in results. (Rolling out bands 3.1)
- The strips to make into bands for rolling out. These can also be copied and enlarged from Figure 1. (Rolling out bands 3.1)
- The strips to make into bands for printing. (Cartridge or cash register receipt roll). These can also be taken from Figure 1 enlarged. Bands must be large enough for a shape punch to used roughly center from edge of band. They must be at least 10 times longer than they are wide. (Printing 3.3)
- Scotch tape or high quality masking tape, 1 per group. (Rolling out bands 3.1 and printing 3.3)
- Large paper sheet such as wallpaper lining paper or cash register receipt roll for each table. (Printing 3.3)
- One or more of three alternative printing methods (Printing 3.3):
    - Shape punches (asymmetric and symmetric see 3.3), sponge make-up wedges, ink pads of one or more colors and disposable gloves.
    - Strong cartridge paper for hand-cut stencils, make-up sponges and ink pads of one or more colors and disposable gloves. Stencils can be cut out from thin plastic document folders.
    - Tracing paper and pencils.
- One or more enlarged copies of Figure 7 per group and scissors. If printing is to be combined with this activity, the tiles must be large enough for the shape punches (Tile placing 3.5).

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