Dances of Heavenly Bodies

Dance, N-body Choreographies, and Change Ringing

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Abstract

Patterns that employ sequences of distinct permutations of dancers are found in dance forms such as English country dance and contra dance, dating at least to the 1600s in England. Contemporaneously in England, the art of church bell change ringing used very similar permutation sequences. More recently, mathematicians and theoretical astrophysicists have discovered that similar patterns are stable orbits for heavenly bodies under the influence of gravity. This paper explores the connections between these art forms and recent mathematical work as well as two dances choreographed by the author that also make use of these patterns.

Introduction. In 1991 our dance company, the Dr. Schaffer and Mr. Stern Dance Ensemble, based in Santa Cruz, California, created an evening length work by Erik Stern and myself based on Melville's short story, "Bartleby, the Scrivener." The story involves a character who works all day copying documents by hand, so we created a number of dances in which the dancers manipulate ordinary sheets of paper. In one sequence the five dancers in the piece move in the four-leaved clover pattern shown in Figure 1. At any given moment only one dancer moves through the center while the others are moving in a flattened curve away and then back towards the center. A number of years later I noticed that a similar periodic pattern – though with the loops not passing directly through the central point - had been discovered for theoretical celestial bodies of equal masses acting under the influence of gravity. In this paper I will point out some connections between the patterns discovered by mathematicians and theoretical astrophysicists, various forms of dance, and the art form of church bell ringing known as change ringing.

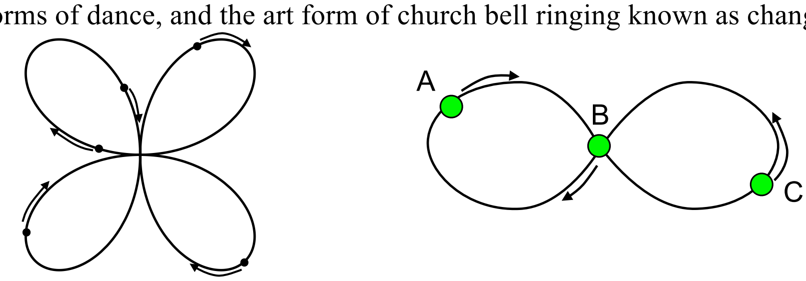

Figure 1: *Dance patterns and solutions to 4-body and 3-body problems*

N-Body Choreographies. In 1993 a solution to the three-body problem in which three equal masses chase each other along a figure eight pattern in the plane, as in Figure 1b, was discovered numerically by Christopher Moore [9], and rediscovered in 2000 by Alain Chenciner and Richard Montgomery [4,7], who gave a rigorous proof of the existence of the figure eight orbit, and expounded on its symmetries. Gareth Roberts proved that the orbit is stable in 2007 [13]. This orbit is much like the "3 person weave" pattern used as an exercise in basketball and other sports, or the cascade pattern using an odd number of balls in juggling. Following these discoveries, other beautifully symmetric planar orbitals have been found involving any number of bodies of equal mass [8]. One simple example resembles the shower pattern in juggling, where any number of balls are tossed from the same hand in a circular pattern, each following the preceding ball exactly. This pattern is also found many circular folk dance routines. A number of non-planar orbitals involving three or more bodies, not necessarily of equal mass, have been found [10, 11], and in 2013, thirteen new solutions to the three body problem were announced [3]. These patterns have been dubbed "n-body choreographies" [4] due to their beautiful dance-like symmetries, but specific connections to dance (or change ringing) do not yet appear to have been described in the literature.

Change-Ringing in Dance. If we label the three bodies – or dancers – in the figure eight pattern in Figure 1(b) as A, B, and C, then they will first line up in that order in a line slanting slightly downward to the right, as shown. They will sequentially give all six permutations in the order shown (vertically) in Figure 2(a), the lines alternately slanting slightly down and up.

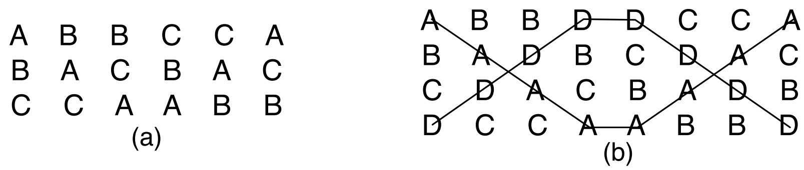

Figure 2: *Permutations found in the figure eight and hey for four patterns*

Figure 3(a) shows a diagram for what is known as a hey for four in contra dance and English country dance (or reel for four in Scottish country dance). This pattern has also been found to be a solution to the 4-body problem, where it has been dubbed "Gerver's super eight," after its discoverer [8].

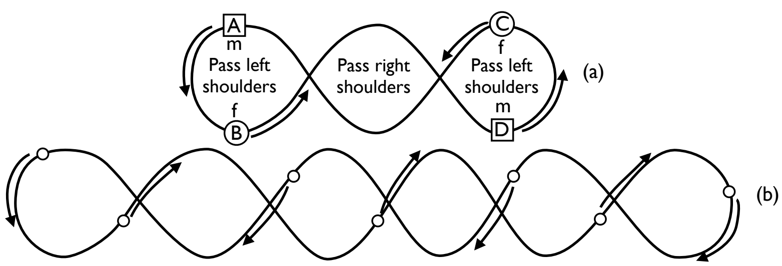

Figure 3: (a) *hey for four* (b) *7-body 6-chain*

Here A and D are the male partners of females B and C, respectively. As they form lines, the dancers repeat the eight permutations, shown in Figure 2(b). This creates one third of the possible 4! = 24 permutations on 4 letters. Note that it is accomplished first by a pair of transposition switching the first with the second letters, at the same switching the third with the fourth, followed by a "central" transposition switching the second with the third letter. This pattern then alternates. The common cycle notation for these switches is (12)(34) followed by (1)(23)(4). Note also that each dancer proceeds alternately to the front and then the back, shown in Figure 2(b) by the lines overlaying the letters A and D. This is also a portion of a common "method" in the art form of change ringing, in which heavy church bells are rung in sequences designed to ring all n! permutations of the *n* bells. Generally such patterns only involve up to 6 or 7 bells, since the factorial function grows faster than exponentially, and 7! = 5040.

Change ringing dates back at least to 1612 in England when the first known company of church bell ringers was established [6]. English country dance was first codified in print by John Playford in his 1651 book on the form [12]. Contra dance was developed from English country dance and French dance forms. Given the similarities in the permutation sequences in change-ringing and these dance forms, plus their origins around the same time and in the same country, it would be interesting to know if there were more specific historical connections between them.

Larry Copes has produced a web site that catalogues common contra dance sequences as both permutations of a square and representation by 2×2 matrices [5]. James V. Blowers describes the permutations of partners in contra dance, relating them to mattress turning and quilt patterns [1]. He illustrates the sequence of permutations of partners in contra dance, with a diagram like that shown in Figure 2b that is almost identical to those showing a "method" used in change ringing. With n couples, this dance takes $2n$ rounds to return all dancers to their starting positions. During a switch in order between two couples, the four dancers form a square and do a short dance sequence together. In this way everyone dances with everyone else at least twice. C.J. Budd and C.J. Sangwin also noted the similarities between country dance and change ringing [2], but point out that because of the distinctions between male and female dancers the dance forms may have more restrictions on the set of allowable permutations. Budd and Sangwin also show a connection between these permutation sequences and knitting patterns. In both dance and knitting, though, it matters which strand or dancer passes in front and which behind, while in change ringing the bells are rung in time rather than space.

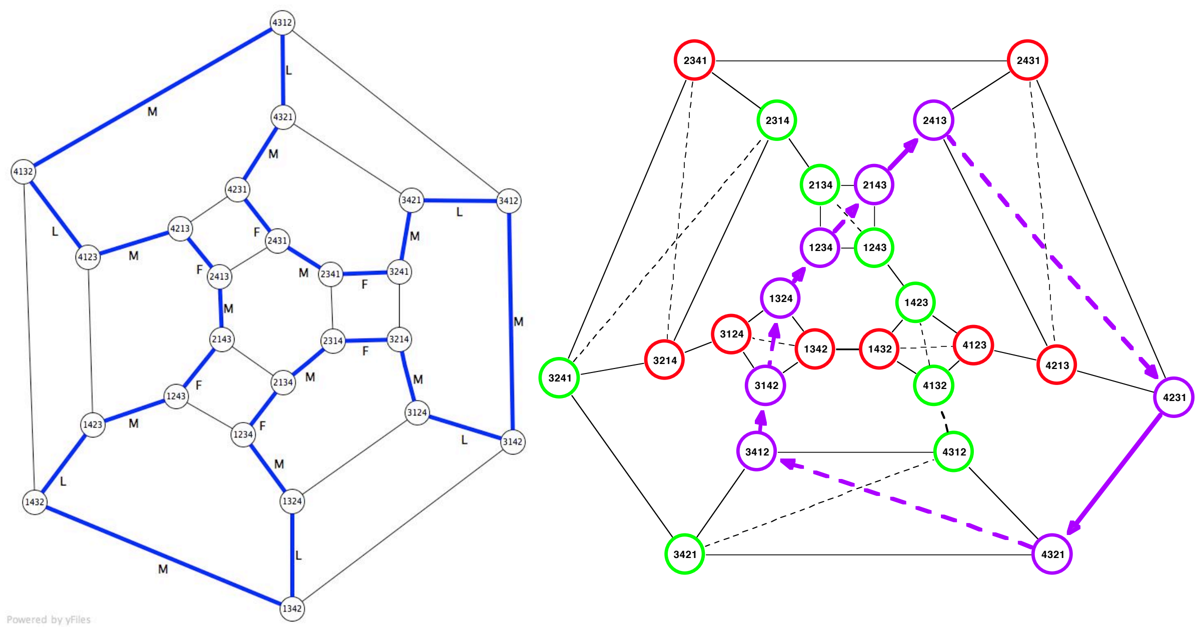

Figure 4: *(a) Hamiltonian cycle using single transpositions, (b) 8-cycle allowing double swaps*

In 2009 I created a dance, "Switch," in which four dancers move through all 24 permutations of themselves in a line, danced to a score of church bell changes. One of the dancers, who is an identical twin, recites humorous stories about her and her sister's childhood escapades with identity switching, while the other dancers comment drolly on aspects of DNA mutations. If we form a graph with 24 vertices representing the 4! permutations, with two vertices adjacent if they differ by one transposition of neighbors, we obtain the "permutahedron," which is also the skeleton of a truncated octahedron, shown in Figure 3(a). In this dance we wanted to use all single transpositions between neighbors to focus audience attention on each switch, so used the Hamiltonian cycle through the permutahedron shown with thicker edges. Here F, M, and L represent swaps of the first, middle, and last dancers, respectively. If we instead allow double swaps as in Figure 2(b), then it is possible to link the 8-cycle shown in Figure 4(b) to two other similar octagons via single swaps ("dodges" in change ringing), and form a commonly used bell pattern for 24 changes known as Plain Bob Minimus.

The octagon in Figure 4(b) gives the same sequence of permutations used in the hey for four and the 4-body choreography. I taught this in a workshop at 2012 Bridges [14], and it was easily learned by a group including non-dancers. N-body choreographies have also been discovered that extend the number of crossovers in hey for four *m* times, for any *m*. Called *m*-chains, these duplicate the pattern of couple swaps in the line of contra dancers. These *m*-chains also often involve various numbers of moving bodies.

The grand right and left in a square dance formation of four couples (or circular heys for four, six, or larger even numbers in English country), in which the men circle in one direction and the women in the other, each dancer alternately passing right and left shoulders with dancers moving the opposite way, has not apparently been found as a solution to the n-body problem. Other English country dances with patterns similar to n-body choreographies are Uffington Horse, which uses two heys for four performed perpendicular to each other; Lull Me Beyond Thee uses other complications that one long-time English country dancer described to me as "like Rubik's Cube," since two lines of four facing each other "horizontally" quickly transform to two lines of four facing each other "vertically."

Questions for Further Investigation. Are there patterns found in contra dance, English country, or other dance forms, or in change ringing that might suggest new solutions to the n-body problem? Were there actual historical connections between those who first formulated or codified English country dance and change ringing? What are the possibilities for creating new dances using n-body choreographies patterns? What other connections might be found between dance, n-body choreographies, juggling, knitting, or change ringing?

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