Constructing and Applying the Fractal Pied de Poule (Houndstooth)

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Abstract

Time is ready for a fractal version of pied de poule; it is almost "in the air". Taking inspiration from the Cantor set, and using the analysis of the classical pattern, we obtain a family of elegant new fractal Pied de Poules. We calculate the fractal dimension and develop an attractive fashion item based on the new pattern, to be showed at Bridges.

1 Introduction

In earlier work we analyzed the mathematics behind the classical Pied de Poule, also called Houndstooth (see Fig. 1) with tools such as tessellation theory, compact Processing programs and compass Logo [1]. In

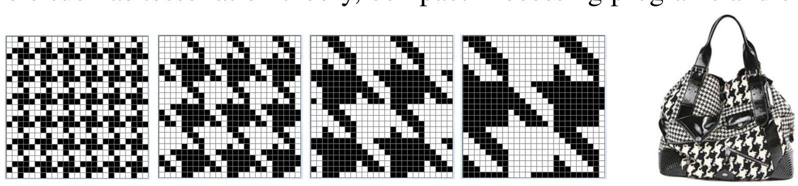

Figure 1: *Successive PDP patterns for N=1,2,3,4 from [1] (a) and McQueen's bag (b).*

this section we shall argue that time is ready for fractal pied de poule. The fashion community is awaiting and subtly announcing this innovation, yet not quite sure how to do it properly.

In Fig. 1 (b) we see Alexander McQueen's 2009 ladies bag with mixed large and small PDPs (pied de poules). Figure 2, (a) is one of Dior's designs of 2012 (there exists a similar design by Neil Barrett for men 2009). The PDP figures are isolated and appear to fly out. In December 2012 we spotted the jacket by Gerry Weber (Fig. 2, b) with almost-PDP figures inside the main-level figures (yet the knitwork has not a precise classical PDP). Even PDP antagonist blogger Anti-Houndstooth saw the possibility coming in 2009: "*Sometimes I wake up in a cold sweat to the idea that one day some slick mathematician will discover the fabric of the universe is a houndstooth weave and the mandelbrot set will reveal universe after universe of pulsing houndstooth patterns!!*" [4]. In 2012 we began searching for fractal pied de poule (fPDP), main results being communicated at the mini-symposium "mathematics and art" of Eurandom, Eindhoven, July 5, 2012 and published now for the first time.

Figure 2: *Christian Dior's 2012 (a) and almost fractal PDP by Gerry Weber 2012 (b).*

2 Towards a fractal

We take inspiration from the well-known Cantor set. This is a subset of the unit interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. The subset is formed by splitting the interval in three segments and removing the middle part $\{x \in \mathbb{R} \mid \frac{1}{3} < x < \frac{2}{3}\}$. This process can be repeated on the two remaining parts, and so on. The set of points *not* removed is the Cantor set. It is a fractal: it is equal to two copies of itself, each shrunk by a factor of 3 and translated [3][1].

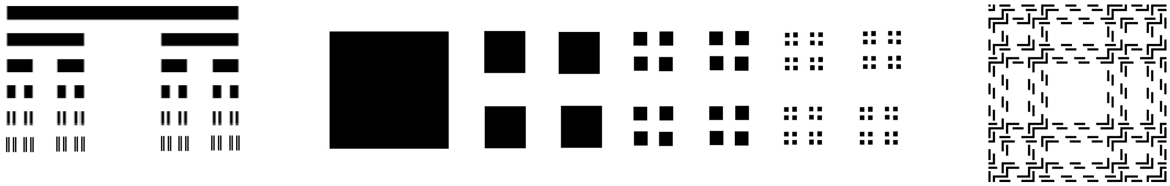

Figure 3: *Approximating Cantor set (a), Cantor dust(b), and twill-woven Cantor-set warp and weft (c).*

The first idea was to replace the black-white pattern of the weft and of the warp by a Cantor set each. Weaving in twill binding we hoped for a fractal PDP. Regretfully this did not work out, we got sparse grids which looked neither fashionable nor PDP-like (Figure 3 c). Then we turned to re-using the results of [1]:

- for each $N > 0$ there is a PDP pattern which can be compactly described by two nested `for` loops with loop-counters i and j and a compact Boolean formula in i and j.
- for each $N > 0$ there is a unique figure consisting of $8N^2$ squares such that an equal number of black and white tiles fit together in a tessellation which is precisely the classical PDP pattern of type N.

An example Boolean formula is $(i-j)\%4 < 2?\, i\%8 < 4 :\ j\%8 < 4$, which produces the second pattern from Fig. 1 ($N = 2$). In Fig. 4 we show the two *basic figures* for $N = 1$ and $N = 2$, alongside two "faux" figures which *do* tile to a PDP, but are not basic since they do not correspond to the most symmetric and compact compass Logo contour description. The basic figures converge towards the *limit figure*.

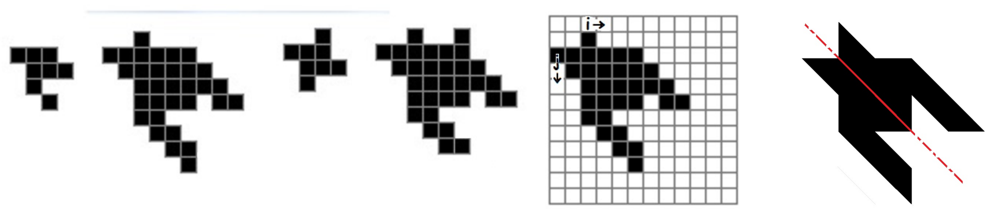

Figure 4: *PDP figures for $N = 1$ (a,b), tiling "faux" figures (c,d), figure b in grid (e), and limit figure (f).*

The new recipe is: take one figure and replace each of its $8N^2$ squares by a scaled down figure, surrounded by an equal amount of white. For scale-down factor $4N$ an elegant fractal pattern arises which visually relates to PDP, and which we call fPDP (fractal PDP). E.g. if $N = 2$, let $\mathtt{e}(i, j)$ for $-2 \le i, j < 10$ be the Boolean function defined by Figure 4 (e), where black means `true`. Consider the recursive program:

```
void  fPDP(float d, float x, float y, float S){
      if (d <= 0)
          rect(x,y,S,S);
          else for (int i = -N; i < 5 * N; i++)
                    for (int j = -N; j < 5 * N; j++)
                         if (e(i,j))
                             fPDP(d - 1, x + i*S / (4*N), y + j*S / (4*N), S / (4*N));
}
```

The d regulates recursion depth and $\mathtt{rect}(x, y, w, h)$ draws a black rectangle of width w and height h. Intuitively the fractal is $\lim_{d\to\infty}\mathscr{B}(\mathrm{fPDP}(d, x, y, S))$ where $\mathscr{B}(f)$ is the closed subset of $\mathbb{R}^2$ marked black by f. In the notation of [5] (Hutchinson's iterated function system) we have a family of $8N^2$ contractions $S_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ so that $|S_{i,j}(x) - S_{i,j}(y)| \le c|x - y|$ with contractivity factor $0 < c < 1$ viz. c=$1/4N$ and hence

[1] The Cantor set has a fractal dimension [5] of $D = \log 2/\log 3 = 0.6309$ and the Cantor dust has $D = \log 4/\log 3 = 1.2619$.

there exists a compact non-empty set F satisfying:

$$F = \bigcup_{-N \leq i,j < 5N}^{e(i,j)=\text{true}} S_{i,j}(F)$$

In practice we work with finite d. The fractal approximated by Figure 5 for $N = 1$ is *uniform* in the sense that the subfigures, sub-subfigures etc. are all of the same N-type. We can make patterns with the N-type increasing for sub-figures, or even differ per position. The results with *de*creasing N are practical, but stopping at or before $N = 1$, formally not fractals.

Figure 5: *Fractal Pied de Poule (fPDP) approximations for* $N = 1$ *(*$d = 2$*,* $d = 3$ *and* $d = 4$ *respectively).*

3 Application

The next step is designing a real fashion item, an elegant men's shirt. The result is shown in Fig. 6. This is a non-uniform fPDP where the main figure has N-type 3, the subfigures have N-type 2, the sub-subfigures have N-type 1 and then the recursion stops. The pattern is generated by the recursive algorithm in Processing, post-processed in Adobe Illustrator and cut with a Speedy300 laser cutter at TU/e. The shirt was designed and welded in the fashion technology studio by-wire.net. The outer layer of white fabric has been laser-cut and the tiny holes reveal the black layer underneath. We will bring the shirt to Bridges and show it in action.

4 Fractal dimension

For uniform fPDP we calculate the fractal dimension using box-counting [5]. E.g. for $N = 2$, if at some zoom-in depth, one figure can be covered by a disc of diameter $\varepsilon = \varepsilon_0$ then at the next level it has 32 sub-figures, each covered by a disc of diameter $\varepsilon = \varepsilon_0/8$. The next level needs $n = 32^2$ discs of diameter $\varepsilon = \varepsilon_0/64$. At depth d it takes $n = n(\varepsilon) = 32^d$ discs of diameter $\varepsilon = \varepsilon_0 8^{-d}$ to cover the fPDP. The fractal dimension is defined as $D = \lim_{\varepsilon \to 0} \frac{\log n(\varepsilon)}{\log 1/\varepsilon}$ so $D_2 = \lim_{\varepsilon_0 8^{-d} \to 0} \frac{32^d}{\log(1/\varepsilon_0 8^{-d})} = \lim_{8^{-d} \to 0} \frac{\log 32^d}{\log 8^d - \log \varepsilon_0} = \lim_{d \to \infty} \frac{\log 32^d}{\log 8^d - \log \varepsilon_0} =$ $\log 32 / \log 8 = 1\frac{2}{3}$=1.6667. For $N = 1$ we find $D_1 = 1.5$, for $N = 3$, $D_3 = 1.7211$. Generally $1\frac{1}{2} \leq D_N < 2$ for $1 \leq N < \infty$ and $\lim_{N \to \infty} D_N = 2$ (i.e. Fig. 4, f, viewed as fractal has infinitesimally shrunk white space inside).

5 Conclusions

It was in the air, now fractal Pied de Poule is a reality. The proposed patterns are new to the best of our knowledge. The work of Fig. 6 will also be shown at the Bridges art exhibition. We contacted the author of Anti-houndstooth [4] and to our happy surprise he was very positive: "*I had long given up hope that there was any chance of redemption for the abused and maligned pattern. I am happy to find that your project*

Figure 6: *Realized fractal shirt. Photo Brian Smeulders, model Stephanie Samson. (©Marina Toeters)*

reinvigorates the iconic motif with energy and purpose." An interesting question is whether the new pattern can still be woven on a traditional loom (we leave that as an option for future research). We thank Chet Bangaru and Jasper Sterk for their support.

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