Girl's Surface

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Abstract

Boy's surface is the simplest and most symmetrical way of making a compact model of the projective plane in R^3 without any singular points. This surface has 3-fold rotational symmetry and a single triple point from which three loops of intersection lines emerge. It turns out that there is a second, homeomorphically different way to model the projective plane with the same set of intersection lines, though it is less symmetrical. There seems to be only one such other structure beside Boy's surface, and it thus has been named Girl's surface. This alternative, finite, smooth model of the projective plane seems to be virtually unknown, and the purpose of this paper is to introduce it and make it understandable to a much wider audience. To do so, we will focus on the construction of the most symmetrical Möbius band with a circular boundary and with an internal surface patch with the intersection line structure specified above. This geometry defines a Girl's cap with C_2 front-to-back symmetry.

Introduction

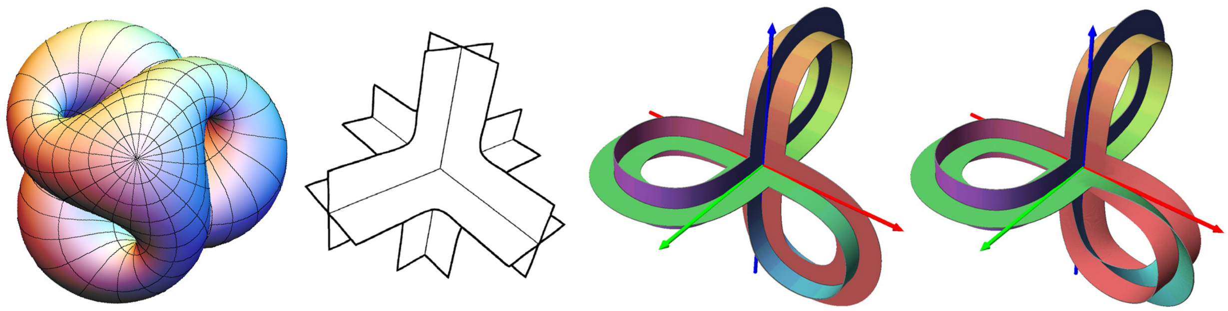
In 1901, when Werner Boy found a way to immerse the projective plane smoothly in 3-space, with no creases or corners or singularities, it took the math world, including his advisor David Hilbert, completely by surprise. Some models of the projective plane, such as the cross cap surface or Steiner's Roman surface (Fig.3, [10]), had been known for some time, but they had singular points with infinitely high curvature (Whitney umbrellas). It was widely believed that some singularities are unavoidable. Boy saw that they are not. Boy's drawings of the surface are impressive (equations took another 75 years to find); however, the surface remains a challenge to visualize. This surface comes in two mirror-image variations that cannot be smoothly transformed into one another via a regular homotopy. Figure 1a shows what we call the right-handed version, because when we puncture this surface it becomes regular homotopically equivalent to a right-twisting Möbius band.

But could there be another way of smoothly immersing the projective plane in 3-space with such a simple intersection set? Or did Boy find the only way? Certainly none seemed to be known to experts in the field. And it is known [8] that if we allow the surface to pass through itself, creating more – or different – self-intersections, then any projective plane in 3-space is smoothly deformable to Boy's surface or its mirror image. Hence it seems reasonable to expect there to be no others with this intersection set.

In attempting to prove that was the case, Goodman and Kossowski [4], surprisingly, happened upon another immersion with quite different properties. While it is now clear that Apéry knew of it and refers to it in his lovely book *Models of the Real Projective Plane* [1], it had gone unnoticed in the mathematical community, perhaps because there was not a smooth model of it. It is this alternate surface – dubbed *Girl's surface* – that we explore here.

Models of the Projective Plane

Any model of the projective plane in 3-dimensional space must intersect itself [9]. It is the structure of the intersection lines that we focus on. We start from the simplest possible intersection set for a smooth model of the projective plane in 3-dimensional space. It is known [2] that this intersection set must comprise a triple point: a point that locally appears as the intersection of three planes (Fig.1b). Six intersection-line branches emanate from this point. They form three loops circling back to the triple point. If we do this in the simplest and most symmetrical way (Fig.1c), we obtain the complete intersection set for Boy's surface. Now the question arises: Could there be another model of the projective plane with a connected intersection set and only one triple point that is homeomorphically distinct from Boy's surface, which means that there is not a one-to-one correspondence between the two models for every surface region bounded by an intersection line? The answer is *yes*, but there is only *one* other way: Girl's surface. It turns out that for this surface, the neighborhood along one of the three intersection loops is twisted through 180° (Fig.1d).



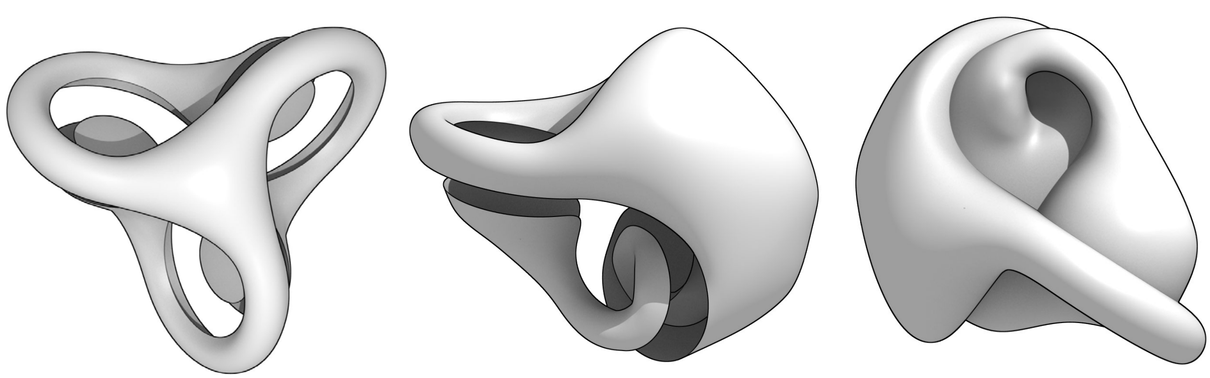
(a) (b) (c) (d)

Figure 1: *(a) r-Boy Surface; (b) triple point; (c) r-Boy and (d) l-Girl intersection-line neighborhoods.*

Constructing Girl's Surface – Starting from the Intersection-line Structure

Tracing around the rims in Figure 1c, we find four complete circuits that close back onto themselves. The three of them that lie inside the intersection-line lobes can be readily filled in with disks. The fourth one winds around the outside in a rather complicated fashion, but it too can be extended and capped off without introducing any new self-intersections (Fig.2a). The result is Boy's surface. This is almost the reverse of Séquin's construction (Fig.5 [10]), which results in the creation of a triple point by filling in a circular hole with opposite points identified – as opposed to starting with the triple point and working outwards along the intersection lines.

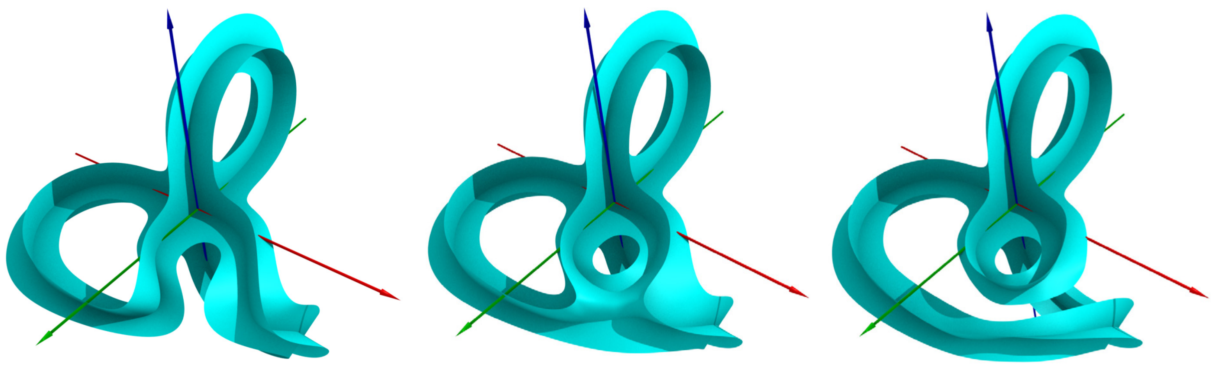
But what happens if we connect one or more of the intersection-line lobes with a half-twist back to the triple point as shown in Figure 1d for the bottom lobe? Tracing around the rim of this figure, we can see that there are still four complete circuits; two of which (the two upper ears) can still be readily capped off with local membranes. The other two circuits present more of a challenge. However, it is still possible, as indicated in Figure 2b. And the result is Girl's surface (Fig.2c). It is homeomorphically distinct from Boy's surface: There, each of the three intersection-line loops bounds a simple disk on the surface. This is not the case for Girl's surface: The loop with the half-twist does not bound a disk on the surface! The special behavior of this lobe also destroys the 3-fold symmetry of Boy's surface. For more models of Girl's surface, see [6].



(a) (b) (c)

Figure 2: *Capping off the rim circuits: (a) of Fig.1c, (b) of Fig.1d; (c) the resulting l-Girl surface.*

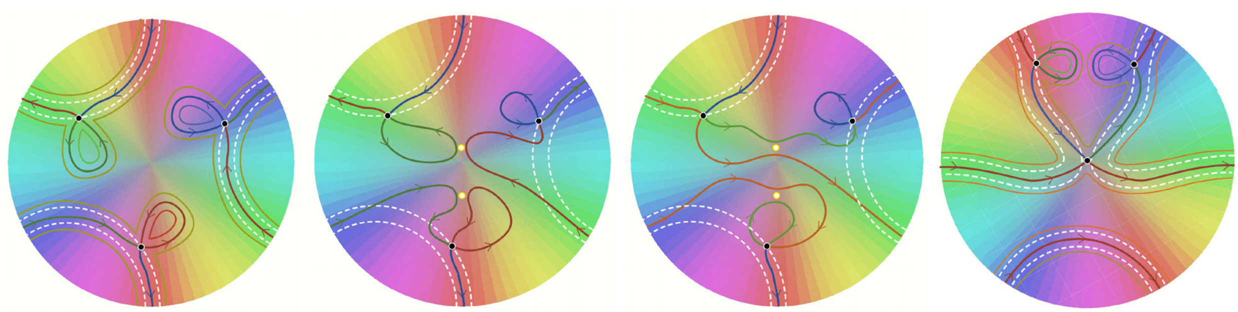
As mentioned above, every smooth model of the projective plane can be transformed by a regular homotopy into either a right-handed version of Boy's surface (Fig.1a) or into its mirror image (l-Boy). So, what might such a deformation sequence look like for Girl's surface? Figure 3 shows a few crucial stills from an animation of the complete transformation of an r-Boy surface into an r-Girl surface. It shows the point where two intersection lines bulge out and touch and form two new circuits – a small local one, and large, contorted one combining the remnants of the two merging loops.



(a) (b) (c)

Figure 3: *Deformation of the intersection neighborhood from Boy's surface to Girl's surface*

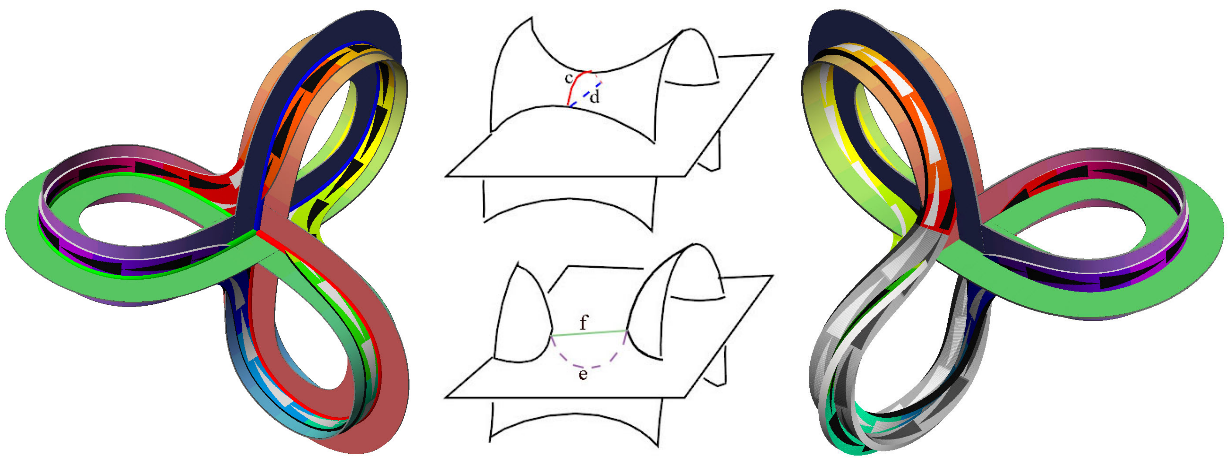
This merging and recombination of intersection loops is shown schematically on the domain map of the projective plane (Fig.4a-d), which can be seen as a disk on which opposite points are identified (i.e., connected by going through infinity). The rainbow coloring shows this implied connectivity and indicates which perimeter points are connected to one another. The darker lines on these diagrams mark the three intersection-line loops, color coded to match the loopy tubes in Figure 5a. Every intersection line element shows up twice, because two different points of the surface meet in those lines. The triple point, marked with a black dot shows up three times. Also shown in these diagrams in white dashed outlines is a Möbius band embedded in the Boy surface. It mostly follows the intersection lines, and it almost touches the triple point three times. This Möbius band is also shown in the intersection-line neighborhood model (Fig.5a). As shown, it makes three left-handed 180° flips, which makes it regular homotopically equivalent to a simple right-twisting Möbius band; this shows that we are dealing here with an r-Boy surface.



(a) (b) (c) (d)

Figure 4: *Intersection-line loops shown on the domain map of the projective plane: (a) Boy surface; (b) before and (c) after the merger of the red and green loops; (d) cleaned-up pattern for Girl surface. (To know what orientation of surface this represents, one would have to specify in what direction the Möbius flips occur as one steps across the perimeter.)*

This topological change is made possible by letting one surface region pass through a saddle shape formed by a different surface region, as schematically indicated in Figure 5b. Once the intersection neighborhoods have separated, we have again three loops that can be simplified and forced into a more symmetrical configuration. However, it turns out that one of the newly formed loops (the bottom one) now has 180° twist in it. Moreover, the chirality of this configuration has changed (Fig.5c) and the three "propeller blades" of Figure 1c are now twisted in the opposite sense! Thus the result of the described regular homotopy move is a mirror image of what is shown in Figure 1d. In other words, when we cut and twist one of the intersection lobes as indicated in Figure 1d, and then complete the surface by capping off all circuits with individual disks, we obtain a left-handed Girl surface, which could then be transformed through the regular homotopy move described above into a left-handed Boy surface.



(a) (b) (c)

Figure 5: *(a) r-Boy intersection neighborhood with embedded Möbius band. (b) Detail of the saddle switch-over. (c) r-Girl intersection neighborhood with embedded Möbius band; (the neighborhood bands of the red loop have been eliminated to better show the two branches of the Möbius band in that loop).*

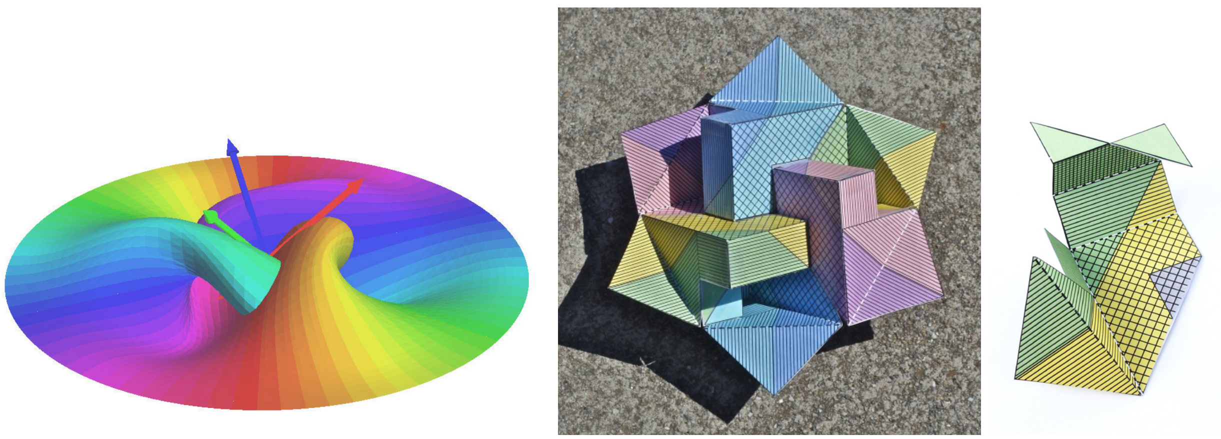
It is natural to ask: What will happen if in Figure 1d we twist not just one lobe through 180°, but perhaps two, or all three of them – or if we apply the loop-combining, regular homotopy move shown in Figure 3 to the intersection neighborhood more than once? When brutally twisting intersection-line loops, we first have to wonder whether we still obtain models of the projective plane, and, if the answer is yes, what the resulting intersection-line structure looks like. When applying additional regular homotopy moves, we are at least sure that we maintain a valid model for the projective plane; we then only have to investigate whether additional intersection-line loops, and perhaps even additional triple points, will be generated.

The full discussion of all these issues is beyond the scope of this paper. Answers can be found in [4] and [5]. The argument that Girl's surface is the only alternative emerges from of a case-by-case study of other possibilities to connect the arcs in Figure 1b. The basic approach is as follows. A formula of Izumiya-Marar [6] tells us that the Euler characteristic of any smooth space model of the projective plane with a single triple point is: χ (projective plane) + # triple points = 1 + 1 = 2. Taking the triple point as one vertex and the intersection-line arcs as three edges, we see that the remainder of the surface must be four disk faces so that $\chi = V - E + F = 1 - 3 + 4 = 2$. Therefore, after the arcs are connected, the rim of the neighborhoods must consist of exactly four complete circuits, and we must be able to cap off each circuit with a disk in such a way that we create no new self-intersections of the surface. Each case is examined: whether opposite or adjacent arcs are connected, whether the connections are made with or without twists, and with or without knots. Only by connecting the arcs as in Boy's surface or as described above for Girl's surface, will we get exactly four complete circuits on the rim that can be capped off by disks without creating any new self-intersections.

Open Symmetrical Models of Boy Cap and Girl Cap

If we puncture Boy's surface (Fig.1a) at its center of 3-fold symmetry and then open that hole into a large circular rim, we obtain a *Boy cap*, which is topologically equivalent to a Möbius band. If we expand that rim into a large "equator" and then distribute the six tunnel entrances symmetrically above and below this equatorial plane, we can obtain a surface with 6-fold D_3 symmetry, where the three C_2 rotation axes lie in that equatorial plane (Fig.6a). This open structure has the advantage that there are no features hiding inside a closed shell. Every part of this surface is clearly visible from one side or the other; thus it is easier to understand.

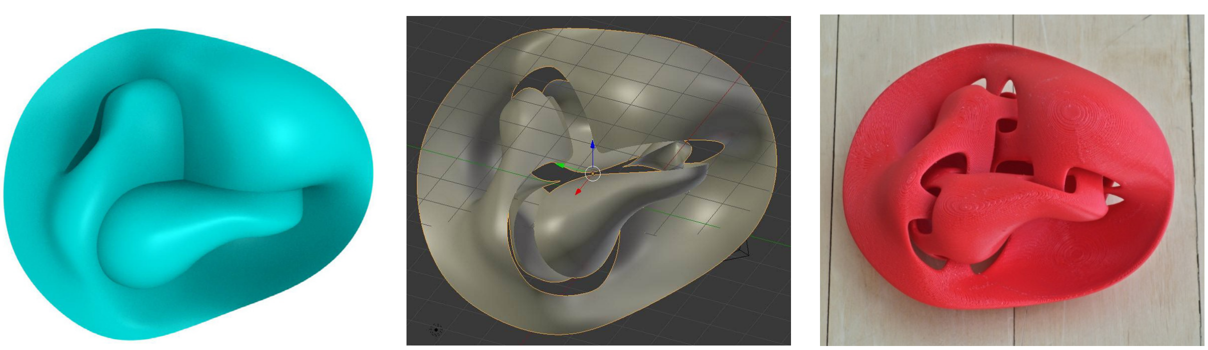
Understanding of such a surface can be enhanced even further by making a paper model. To make this task as easy as possible, we have designed a "cubist" version in which most vertices lie on an integer grid, and most faces join with 90° angles (Fig.6b). A discrete 6-band rainbow coloring has been applied, so that it is easy to see how opposite points on the equatorial circle connect to one another via a helical path through the tunnels of the Boy cap. The whole paper model has been constructed from six copies of the template shown in Figure 6c. These templates will be provided in the form of auxiliary material with the on-line proceedings.



(a) (b) (c)

Figure 6: *Symmetrical, open-rim r-Boy cap: (a) virtual B-spline model, (b) paper model, (c) template.*

An open model is even more valuable for Girl's surface, so that the special twisted lobe can be inspected from both sides. Such a Girl cap is depicted in different ways in Figures 7a-c. The 3-fold symmetry has been broken because there is only one twisted intersection loop. However, we can at least preserve one of the C_2 symmetry axes – the one that goes through that twisted lobe.



(a) (b) (c)

Figure 7: *Symmetrical, open-rim Girl cap: (a) virtual subdivision model, (b) with intersection-line neighborhood removed, (c) a physical model made on a Fused Deposition Modeling machine.*

We are also in the process of trying to make a "cubist" paper model for a Girl cap. However, we find it rather difficult to capture the tight helical twisting of the surface, as it squeezes through the twisted intersection loop, using only 90°-angular facets; it leads to a whole lot of un-attractive and confusing "stair-casing." Thus, making good visualization models is clearly an art. On the other hand, successfully realized models often emerge as art objects in their own right. We continue our efforts to come up with an easy-to-build and attractive paper model for the fascinating, but somewhat elusive Girl surface.

Summary and Conclusion

There are two homeomorphically different compact smooth models of the projective plane that have only a single triple point and a connected set of intersection lines. They fall into four different ambient isotopy classes. Figure 8 summarizes how they relate to one another. It is a challenging mental exercise to try to understand the connectivity of these surfaces and the transformations that deform one into another.

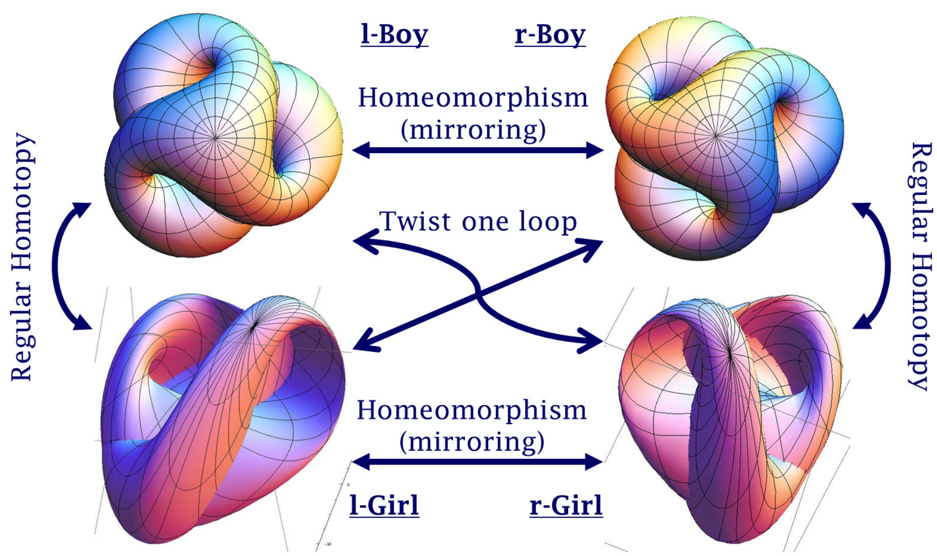


Figure 8: *Different simple, smooth, compact models of the projective plane and their relationships.*

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