Wallpaper Designs of Mirror Curves Inspired by African Sona

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Abstract

The Cokwe people of Africa developed a drawing technique that creates monolinear curves (Eulerian circuits) within a grid of dots where the curves are both symmetric and follow tightly constrained rules. Inspired by these designs, including some that contain wallpaper designs, we search for monolinear curves that abide by the Chokwe drawing constraints and which exhibit the 12 different wallpaper symmetry groups with rectangular translation lattices. In particular, we search for families of such curves that remain monolinear for arbitrarily large rectangles. We show that such families exist on sets of $n \times m$ rectangles of positive density among the set of all rectangles.

1. Introduction

In past generations, when the Chokwe (Tshokwe/Cokwe) and neighboring peoples in eastern Angola and northwestern Zambia would gather at meeting places, storytellers would draw a continuous path in the sand to illustrate their story. The drawings, called *lusona* (singular) or *sona* (plural) accompanied fables, proverbs, games, riddles, and animal descriptions. These are often complex closed curved paths that can be traced in a continuous monolinear motion, where the path encloses each point in a square lattice of points set in the sand by the storyteller preparing the story. Mathematician Paulus Gerdes has studied sona for over 25 years, and shared many of their interesting mathematical properties in his publications. He uses the term *mirror curve* for a wide-spread class of sona whose curves obey the law of reflection (angle of incidence equals angle of reflection) when they encounter the bounding rectangle or a "mirror wall" placed within the lattice of points. [4], [5], [6], [7].

In examining the traditional sona, and other designs of Gerdes in the sona tradition, I noticed many that contained repeating patterns. Earlier, I investigated strip patterns of these designs (see [2]); here I look at traditional sona and invented sona that contain periodic plane patterns. The question I investigate is whether one can find families of sona that essentially contain a given plane pattern for arbitrarily large sizes of rectangles. Such designs cannot contain 3-fold or 6-fold rotations, hence we are limited to the 12 "rectangular" symmetry groups: *p1*, *p2*, *pg*, *pm*, *cm*, *pgg*, *pmm*, *cmm*, *pmg*, *p4*, *p4g*, and *p4m*. [11]

2. An Algorithm for Producing Sona

Gerdes devised the following algorithm to produce sona, based on a mnemonic device we believe was used by the Chokwe sona drawing masters. Begin with a $2m \times 2n$ rectangular grid of unit squares, and place a dot at the center of each of the mn 2×2 sub-squares, as shown in Figure 1a for a 6×10 rectangle. Then place 2-unit long line segments on edges of some of the 2×2 sub-squares. These segments will act as 2-sided mirror walls; the bounding rectangle is also a mirror wall. Begin a rectilinear path (such as traced by a light ray) at the midpoint of an (outer) edge of one of the 2×2 sub-squares, traveling in a direction that makes a 45° angle with the sides of the rectangle. Each time the path encounters a mirror wall it bounces (reflects) and continues until it produces a continuous closed path, as shown in Figure 1b.

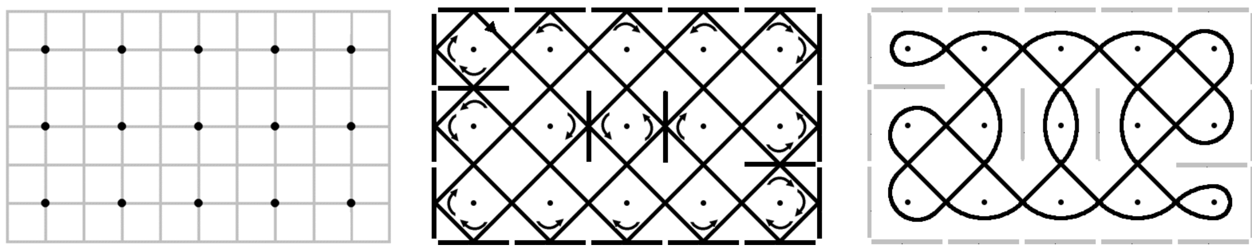


Figure 1: *Views of a mirror curve in a 3 x 5 rectangle.* (a) *The square grid, which we imagine, and the dot layout, which are placed by the Chokwe artist.* (b) *Both the bounding and internal mirror walls, off which the curve bounces, following the arrows.* (c) *The monolinear curve (in black) drawn by a Chokwe artist.*

The mirror curve in Figure 1b is a *monolinear* path in which every unit square in the grid is entered and exited exactly once, and each dot is completely enclosed by the path. The Chokwe artists do not draw rectilinear paths, but instead trace graceful paths following that route, but replacing sharp 90° corners by curves. The lusona in Figure 1c is equivalent to that in 1b, but needs no arrows since the locations of the mirror walls and the bounding walls are clear from the curve itself (they are not drawn by the artist). In this path, every dot is completely "embraced" by the crossing curves of the path.

Chokwe artists used a variety of dot layouts, but we consider only these "rectangular mirror curves." The size of the lattice of dots and the placement of the mirror walls determines the number of components (connected closed paths) in a full mirror curve, that is, one that completely encloses each dot. While some Chokwe sona use more than a single component, Gerdes [7] has shown that the majority of their sona are monolinear, and will be our focus here. Bain [1] describes a related construction process for Celtic knots, which can be interpreted as placing mirror walls from one dot to another. Chokwe art occasionally used such mirror walls as well, but the large majority of their designs use walls placed as shown here, and we limit ourselves to these types of mirror curves, which Gerdes [4] calls "regular mirror curves".

3. Wallpaper Patterns from Sona

It is well known that the mirror curve of size $n \times m$ with no interior mirrors will consist of gcd(n, m) components. Thus the 3 x 5 mirror curve of this type (Figure 2) is monolinear, and its interior shows the familiar *p4m* wallpaper pattern of squares. Since this design can be repeated with larger and larger dimensions of rectangles that are still monolinear curves, it is natural to think of the full family of such curves as *representing* a *p4m* design of monolinear mirror curves. Such rectangles are common, and we wish to formalize their frequency for comparison with other families of rectangles representing other wallpaper patterns.

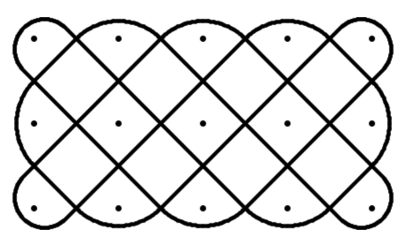


Figure 2: *A 3 x 5 lusona with no interior mirrors; its interior is part of a* p4m *wallpaper pattern.*

If we let R(N) be the set of all $n \times m$ rectangles with $n, m \leq N$, then we can describe the *asymptotic density* of a family of rectangles F as: $\lim_{N \to \infty}\left(|R(N) \cap F| \big/ |R(N)| \right)$. The asymptotic probability that two random numbers less than a fixed N will be relatively prime is $6/\pi^2$, hence the asymptotic density of this family of monolinear sona, representing *p4m*, is about 61%. This then sets a goal of our investigations:

To find families of rectangular mirror curves (with some fixed pattern of mirrors) whose interior design is part of a wallpaper pattern, such that the asymptotic density of the monolinear rectangular curves within all such rectangular mirror curves is positive.

The traditional Chokwe sona known as a "Chased Chicken" (Figure 3) have been recorded in at least three sizes: 5 x 6; 7 x 8; and 9 x 10. [3][9][10]. Figure 3 shows one of size 7 x 8; its interior is clearly part of a *p2* wallpaper pattern. Gerdes has shown that the Chased Chicken family of sona are monolinear mirror curves when they have size (2m+1) x 2n, and gcd(m+1, n+1) = 1. This provides a family of monolinear mirror curves with symmetry group *p2* and asymptotic density of 15.2%. (Figure 3 is not actually monolinear, probably due to an error. Gerdes [7] reconstructs what was probably the original sona.)

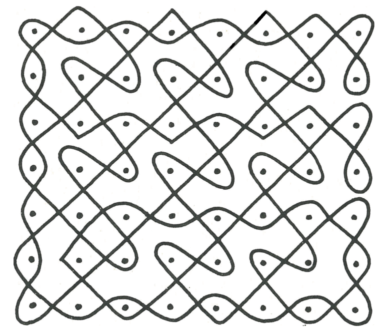


Figure 3: *The traditional "Chased Chicken" sona represent a p2 wallpaper pattern. From [1].*

Rectangular grids, such as the dot layouts used for these sona, cannot have rotations points of 1/3 or 1/6. Thus only 12 of the 17 wallpaper groups, listed in the introduction, can be represented by families of sona. These previously classified families of sona represent two of these wallpaper groups, leaving 10 more to investigate.

4. The Pumping Lemma for Mirror Curves

Empirically, there seem to be several families of sona representing the various symmetry groups. Finding examples where we can prove which sizes work is more challenging. One effective tool is the lemma below. Gerdes [8] discovered this result independently in a "Circles of Interest" workshop in Africa.

Given a mirror curve, we imagine slicing it into parallel strips created by cutting it along lines of edges of the square cells. The curve *S* of Figure 4, for example, has been cut into the vertical pieces A, B, C, and D by the three dashed lines shown, and we write this as *S* = ABCD. In some cases, we can take pairs of strips of the same height and combine them to create either a larger strip or a full mirror curve. In combining strips X and Y to form a larger strip XY, the right side of X and the left side of Y must reflect from the same mirror walls. In Figure 4, all 3 cutting lines hit mirror walls at exactly the same height, hence we can cut out strips B and C and combine them in various ways with strips A and D. Under certain circumstances, it is possible to guarantee that combinations of these strips will produce monolinear mirror curves.

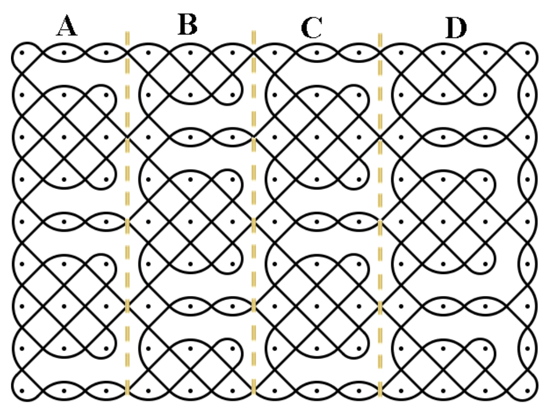


Figure 4: *A monolinear mirror curve that represents symmetry group* cm*. The union of strips B and C meet the conditions of the Pumping Lemma.*

The Pumping Lemma: Let *S* be a *k*-component mirror curve cut into three strips A, B, and C by the two parallel lines L_1 and L_2, so that *S* = ABC, and where:

1. The mirror walls of S that lie on L_1 and L_2 are at exactly the same heights;
2. No line of *S* is completely contained inside strip B; and
3. Every line of *S* that enters B on the left exits it on the right at the same height, and traveling in the same direction, i.e. either both lines are directed 45° up (to the right) or 45° down (to the right).

Then all curves AB^iC are also *k*-component mirror curves, for any $i \geq 0$, where B^i denotes *i* copies of B.

We call this the "pumping lemma" (after a similarly named theorem in computer science) because copies of B can be "pumped in" to the inside of the sona, or the single original strip can be "pumped out".

Proof: By condition 1, the number of lines coming into B on the left must be exactly the same as the number leaving B on the right. With condition 3, this means that every line in B that meets the right edge of B must have come from an edge that entered B on the left. From this we can conclude that there can be no line of S_i completely contained inside the sequence of strips B^i. If there were, then that line would have a left-most strip B_j through which it traveled, and hence by condition 2 it would also travel through strip B_{j+1}, which would violate our second sentence.

Now each line of S that enters B on the left at height h also exits it at height h (see Figure 5 for some examples). Continuing through copies of B, that line would enter the left-most edge of B^i at height h and leave the full B^i on the right at the same height. Thus regardless of the number of copies of B (including none), lines in A and C will be connected in the same ways, in exactly the same order. Thus these lines, and their connecting segments through B^i, will create the same number of total components as in S. By the previous paragraph, there can be no lines of S_i not included in this count, hence S_i is a k-component mirror curve. ■

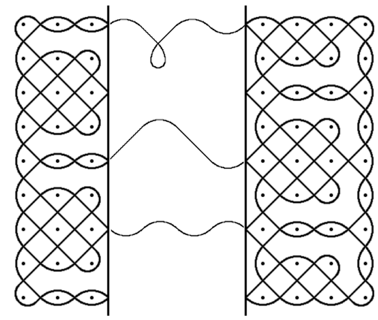


Figure 5: *The conditions for the pumping lemma to apply.*

We can use this lemma to analyze several families of mirror curves. We first verify a conjecture of Gerdes [7] on the traditional sona design "The Lion's Stomach". These sona exist for any rectangle with an odd number of columns of dots and at least two rows of dots. Gerdes conjectured that if a "Lion's Stomach" lusona has width $n = 2k + 1$ and height m, then the number of components of the lusona curve will be 1 when k is even, and m when k is odd. We show this is true.

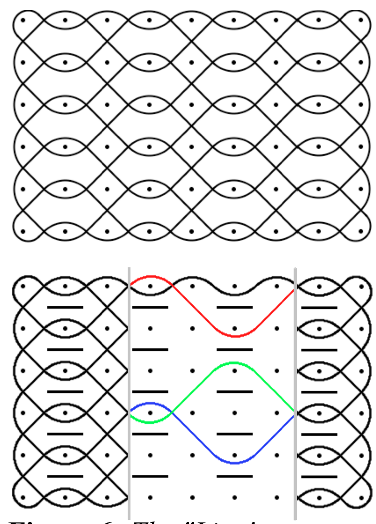


Figure 6: *The "Lion's Stomach" design satisfies the conditions of the Pumping Lemma with a strip of width 4.*

Figure 6 shows the 6 x 9 Lion's Stomach lusona in two views. The top shows the lusona without the mirror walls. The bottom shows the lusona with the mirror walls that determine the drawing of the design, and we show the removal of a central strip of width 4 that meets the requirements of the Pumping Lemma. For ease of verification, we have shown only four of the curves that join the remaining left and right strip; all of the undrawn curves can be seen (from the positions of the mirror walls) to be equivalent to one of these. The Pumping Lemma tells us that we can remove (or add) such strips from this lusona without changing the number of components in the design. More formally:

Corollary: If S is a Lion's Stomach lusona of width $n = 4k + 1$ (respectively 4k+3) and height m, then it has the same number of components as a Lion's Stomach lusona of width 5 (resp. 3) and height m.

Thus we need only understand Lion's Stomach sona for widths 3 and 5, to understand all such sona. We show the 5 x 3 and 5 x 5 Lion's Stomach in Figure 7. It is easy to see that the pattern of walls in the m x 3 Lion's Stomach (Figure 7a) requires m curves to be drawn, thus demonstrating this part of the conjecture. The 5 x 5 lusona (Figure 7b) is monolinear. Figure 7c shows a segment of this curve in thick green. It is the only portion of the curve that traverses the bottom row of dots. If we replace this green segment with the one in Figure 7d, which starts and ends in the same places as the removed segment, we have drawn the 4 x 5 Lion's Stomach lusona having the same number of components (i.e. one). Thus we have proved Gerdes' conjecture:

Theorem: A "Lion's Stomach" sona of width 4k+1 and any height, is monolinear. A "Lion's Stomach" lusona of width 4k + 3 and height *m* has *m* components.

This gives a family of monolinear mirror curves representing *pmm*, with asymptotic density 25%.

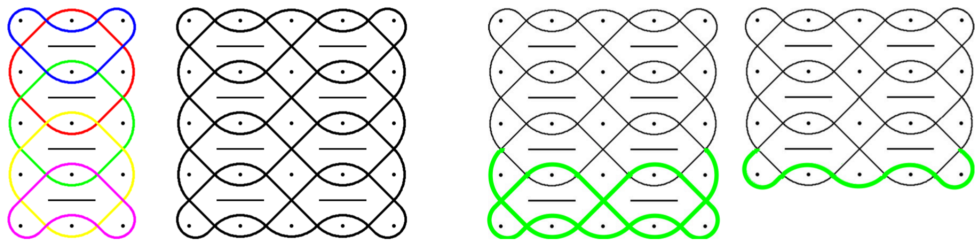


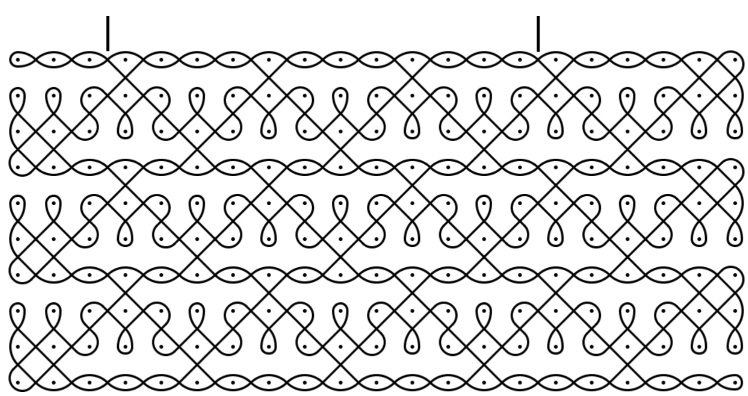
Figure 7: *On the left are Lion's Stomach sona of size 5* x *3 and 5* x *5. On the right is the reduction of the 5* x *5 Lion's Stomach design to the 4* x *5 design.*

5. New Families of Monolinear Wallpaper Mirror Curves

We begin by showing some families of mirror curves where the dimensions of rectangles that have monolinear curves can be determined directly from the pumping lemma. In all cases, the pumping lemma is applied to the strip indicated by lines above the design. This reduces the analysis to that of a single strip (or 2). In some cases, those strips can be reduced from one end, as in the previous section, by a further application of the pumping lemma on a vertical strip of the design, or by other individualized analysis. Generally, once the two-dimensional pattern has been reduced to a single fixed-width strip, the remainder of the proof is straight-forward. We omit the details of the proofs for these individual strips.

The 9 x 13 mirror curve that illustrated the explanation of the pumping lemma (see Figures 4 and 5) is one of a family of monolinear mirror curves for sizes (4n+1) x (12m+1) and (4n+1) x (12m+9). This family represents symmetry group *cm*, and has asymptotic density 4.2%. Figures 8-10 show families of curves that represent the wallpaper patterns *pmg*, *cmm*, and *pm*, with asymptotic densities 8.3%, 8.3%, and 4.2%.

Figure 8: *A family with symmetry group pmg, with pumping lemma strip of width 12. These are monolinear mirror curves for dimensions* (2n+1) x (6m+1). *Since the strip to which the pumping lemma is being applied has width 12, we start this construction with two base cases:* (2n+1) x 7 *and* (2n+1) x 13.



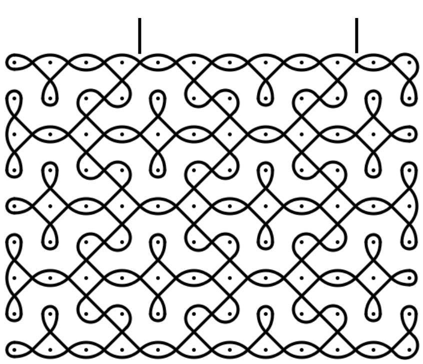


Figure 9: *A family with symmetry group cmm, with pumping lemma strip of width 6. These are monolinear mirror curves for dimensions* (2n+1) x 6m.

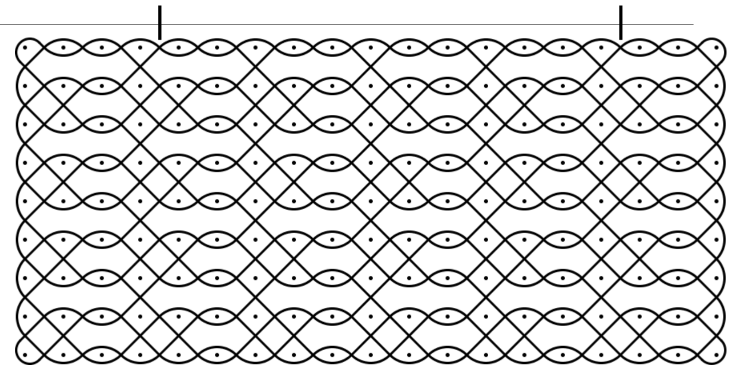


Figure 10: *A family with symmetry group* pm*, with pumping lemma strip of width 12. These are monolinear mirror curves for dimensions* (2n+1) x (12m+7).

We can strengthen our use of the pumping lemma by modifying the designs being analyzed. If a "loop" surrounding a single dot in the mirror curve is cut off (and discarded) by adding an additional mirror at the location where that loop crosses itself, we have a curve that no longer embraces every dot in the rectangle. Nevertheless, the pumping lemma still applies to this curve, and may give us a proper mirror curve. Figure 11a gives a 10 x 9 example of a family of monolinear mirror curves with *p1* symmetry. Figure 11b shows the mirror walls implicit in 11a, along with the mirror walls needed to remove the loops in each module of the curve. Figure 11b also shows the cuts for the pumping lemma, applied horizontally instead of vertically. Figure 11c shows the result of removing all three of those strips. The one region shown converts to the row of mirrors pointed to by those cut lines. The top and bottom rows will also convert into rows of mirrors. This reduction converts a design of this *p1* family of size (3j+1) x (4k+1) into a Lion's Stomach design of size (j+1) x (4k+1). Since all Lion's Stomach designs of this size are monolinear, all of the *p1* designs of the former size are monolinear. This family of sona have asymptotic density 8.3%.

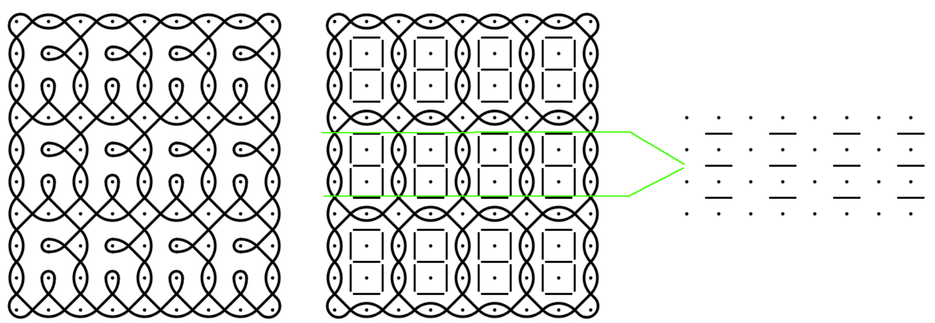


Figure 11: *Removing loops to simplify the application of the pumping lemma. In this case, the pumping lemma is applied horizontally, and is used to reduce this* p1 *design to the Lion's Stomach design solved earlier.*

The argument above applies to *any* arrangement like this where the repeating module of the design has two simple loops embracing those two center points. In particular, the designs of Figure 12 below show designs with symmetry groups *pg*, *pgg*, and *cm*, all generating monolinear mirror curves for rectangles of dimensions (4k+1) x (3j+1), and with asymptotic density 8.3%. (The *cm* pattern repeats a symmetry group from above. But while this design has higher asymptotic density, we believe the earlier example is more aesthetically appealing.)

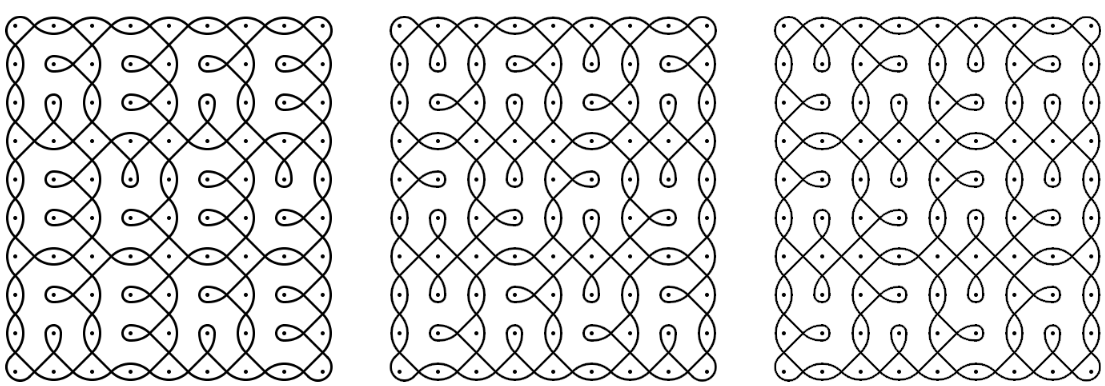


Figure 12: *Families of designs with symmetry groups pg, pgg, and cm respectively.*

6. Four-Fold Symmetry

To provide families of monolinear mirror curves with four-fold symmetry appears to require some additional techniques. Figure 13 shows a 9 x 13 example from a family of *p4g* designs. This family of designs was shown by Gerdes [5], and this analysis appears to be implicit in the work of Gerdes. In Figure 13a, we show the design, with the mirrors hidden. In Figure 14b, we show the placement of the mirrors, with only one-fourth of the drawing completed. It is easy to see from the symmetric placement of the mirrors (a *p4g* pattern itself) that this pattern would continue regardless of the number of columns. This partial drawing continues around the corner, ready to follow the same path through the rows of the design; it repeats twice more to continue around all four sides of the design. More formally, one can prove this by induction. This design will be a monolinear mirror curve for dimensions (4m+1) x (4n+1), with density 6.25%.

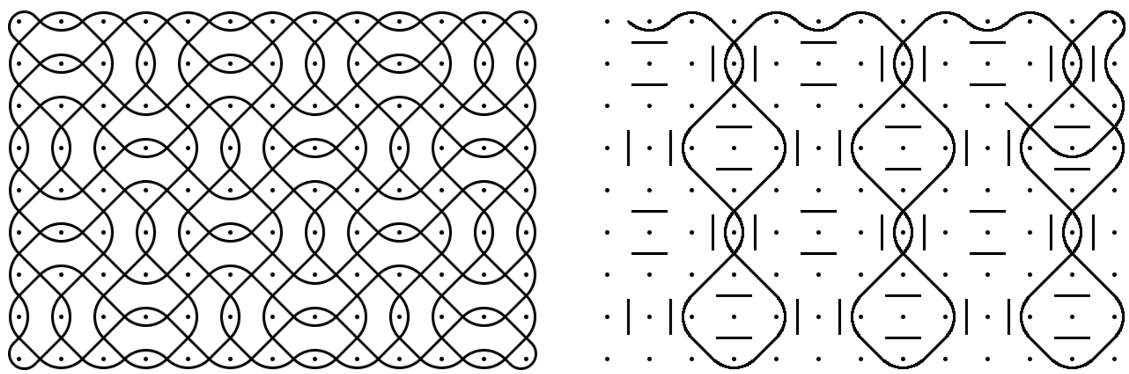


Figure 13: (a) *A family of mirror curves with* p4g *symmetry group. (b) The path for one-fourth of the drawing.*

Figure 14a shows a 15 x 20 example of a monolinear mirror curve with *p4* symmetry. This is constructed from a set of square 5 x 5 modular pieces each with a central 4-fold rotation. These modules are connected at the midpoints of the edges of their bounding squares, creating additional 4-fold rotations centers at the corners where four modules meet. To understand when a rectangle built this way will be a monolinear curve, Figure 14b shows the replacement of three of these modules (one at an edge, one at a corner, and one in the interior of the design) by a more direct "shortcut" that goes to the same connection location on the boundary of the module. If all of the modules are replaced in this manner, we have exactly the same path as the "no mirrors" curve in Figure 2. Consequently, a mirror curve of this form of size 5*n* x 5*m*, will be monolinear only when the "no mirror" rectangle of size *n* x *m* is monolinear, that is, when gcd(*n*, *m*) = 1. This then gives us a family of monolinear mirror curves with symmetry group *p4* with asymptotic density 2.4%.

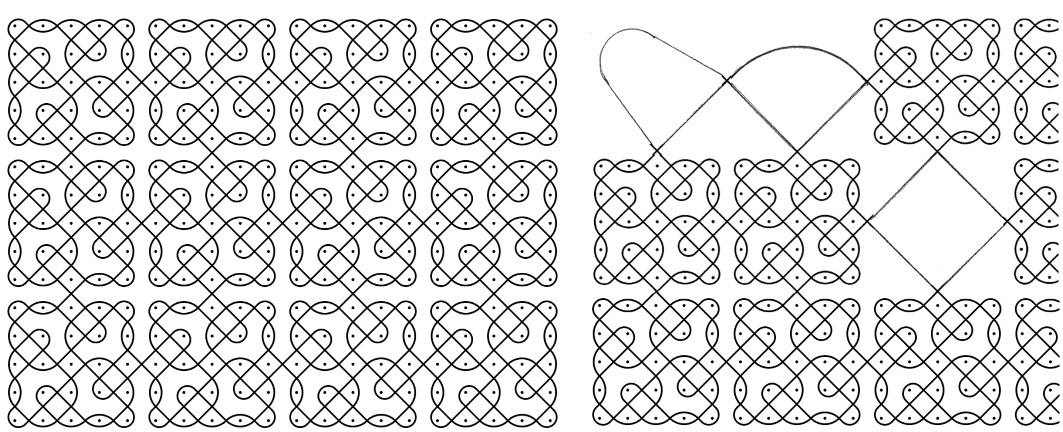


Figure 14: *(a) A 15 x 20 example from a family of mirror curves with p4 symmetry.*
(b) Replacing the 5 x 5 modules with simple curves that join points where the modules connect.

Summary

For each of the 12 wallpaper symmetry groups that can have a rectangular translation lattice, we found families of monolinear mirror curves, of positive asymptotic density within the set of all rectangles of a bounded size, that represent that symmetry. Many of these drawings generate aesthetically pleasing Eulerian circuits. There remains open the possibility of families of sona-like drawings with higher density than discovered here. For example, for *p4* there are likely other families of *p4*-symmetric, monolinear mirror curves with the artistic appeal of that in Figure 14. We suspect that a rigorous investigation of small modules as repetition units, laid out under various symmetry groups, would reveal additional attractive mirror curves. Our investigation here was limited to mirror walls placed on the edges of the internal squares of the bounding rectangle, which is the most common placement among the Chokwe artists. It could be fruitful to investigate the use of mirror walls that are perpendicular bisectors of those edges, as is often implicit in Celtic knots.

References

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