Unfolding Symmetric Fractal Trees

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Abstract

This work shows how the angles and ratios of side to diagonal in the regular polygons generate interesting nested motifs by branching a canonical trunk recursively. The resulting fractal trees add new material to the theory of proportions, and may prove useful to other fields such as tessellations, knots and graphs. I call these families of symmetric fractal trees *harmonic fractal trees* because their limiting elements, i.e., when the polygon is a circle, have the overtones or harmonics of a vibrating string $1/2$, $1/3$, $1/4$, ... as their scaling branch ratios. The term harmonic is also used here to distinguish them from other types of self-contacting symmetric fractal trees that don't have a constantly connected tip set under a three-dimensional unfolding process. Binary harmonic trees represent well-known Lévy and Koch curves, while higher-order harmonic trees provide new families of generalized fractal curves. The maps of the harmonic fractal trees are provided as well as the underlying parametric equations.

Introduction

Branching patterns like those ubiquitously found in nature are aesthetically appealing objects that effectively connect space subsets in a hierarchically ordered manner. Such structures are self-similar at some extent and can be defined by a particular fractal tree. Briefly speaking, a fractal tree is a simple substitution system, i.e. a simple branching rule applied recursively, see Figure 1. In Wolfram's book *A New Kind of Science* [10], simple binary and ternary substitution systems produce a myriad of organic outputs that leave one feeling confronted by an unmanageably complex parameter spaces. But we can still explore a well-defined region of such parameter spaces inhabited by a highly ordered class of symmetric fractal trees.

Here, we are not going to explore the fractal geometry of nature but the inherent fractal nature of geometry. In this sense, regular polygons are exotic fruits from which an amazing fractal flora emerges. An N-gon, a regular polygon of N sides, contains $N-1$ diagonals emanating from a given vertex (including its sides, d_1 and d_{N-1}). Each diagonal d_b in turn engenders a harmonic fractal tree $H(N, b)$ with b branches per node, scaling ratio r equal to the ratio of side to diagonal, and with N-fold rotational symmetry. Then, a dynamical unfolding process will produce its planar dual form, see Figure 3.

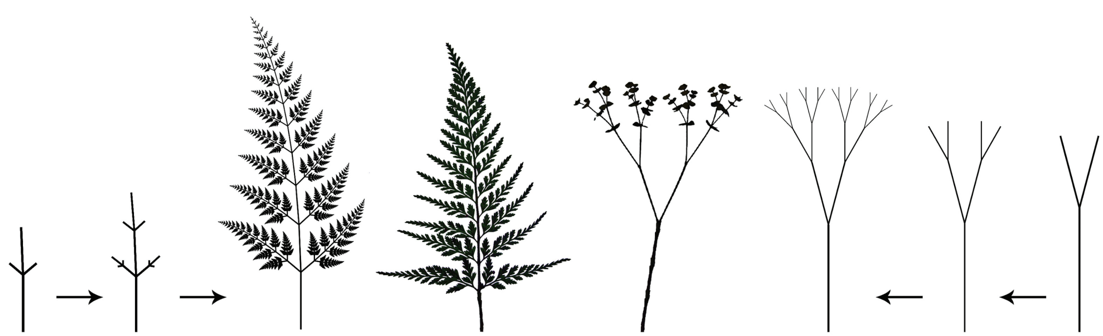

Figure 1 : *A fern and a binary tree-like plant (center) and their characteristic fractal trees (sides).*

The Hexagonal Case

Before going into further mathematical details let's explain how these trees are grown, using the 6-gon (regular hexagon). Associated with each of its five diagonals d_b is a particular fractal tree with b branches per node. Each branch is separated from its neighbor by $\frac{\pi}{3}$ radians with the first left branch placed onto its nearest polygon side as it is shown in Figure 2. For example, the diagonal d_2 that goes from A to C can be seen as the canonical trunk of a symmetric binary tree with a left branch placed from C to B, scaling ratio $r = \frac{CB}{AC} = \frac{1}{\sqrt{3}}$, and a mirrored right branch. This arrangement forms the basic rule of branching that, if applied recursively to the newly born branches infinitely many times, it will eventually generate a fractal tree. This tree can be then assembled around vertex A with five other copies of itself obtaining thus a binary harmonic tree with 6-fold symmetry, see the left-bottom fractal $H(6, 2)$ in Figure 2. It is none other than the tree representation of the famous Koch snowflake, a fractal curve first described in 1904 by the Swedish mathematician Helge Von Koch.

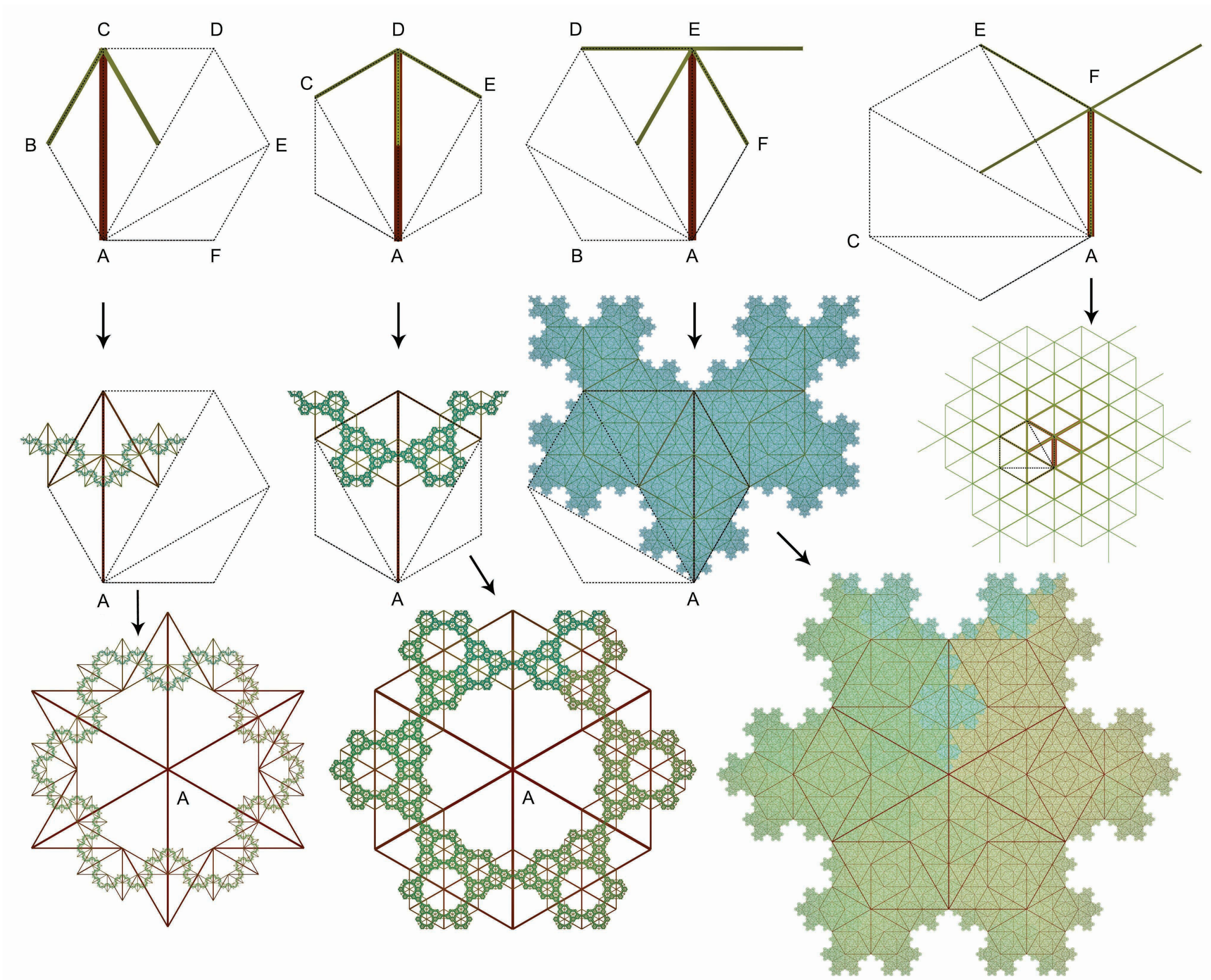

Figure 2 : *Harmonic fractal trees associated to the hexagon, from left to right* $H(6, 2)$*,* $H(6, 3)$*,* $H(6, 4)$*.*

We are now going to make use of a fascinating feature of the Koch curve that was first envisioned by the French mathematician Paul Lévy. In 1938, Lévy published a work entitled *Plane or Space Curves and Surfaces Consisting of Parts Similar to the Whole* [7]. It contained several hand drawings of plane fractal curves, now called Lévy curves, generated by unfolding outwards the smaller triangular components of the

Koch curves, see box A in Figure 3. As he noted, these planar fractal curves were particular cases of a family of three-dimensional Koch-like curves obtained from unfolding the triangular components by a certain angle β at each generation. The 3D family of curves that he introduced can be now obtained by unfolding binary harmonic trees $H(N, 2)$. The unfolding process consist in rotating each pair of newly born branches by an angle β from a particular axis. This axis crosses the common branch node and it is placed in the plane spanned by the mother branch and the symmetrical pair in a direction perpendicular to the mother branch, see the schematic diagram B in Figure 3. When β eventually reaches 180 degrees the harmonic binary tree has unfolded to its dual form, see the left unfolding sequence in Figure 3 C. The unfolding process is easily

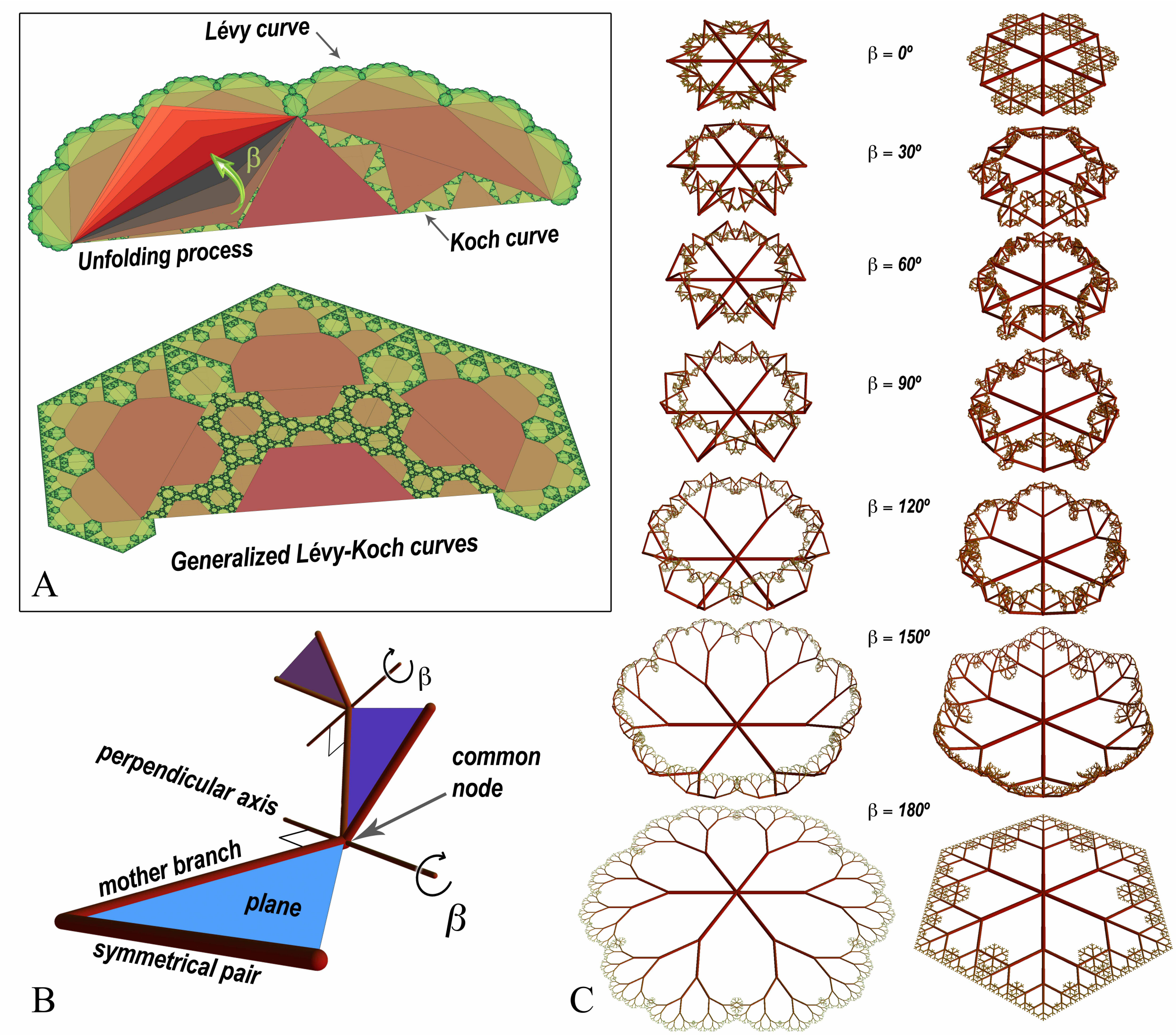

Figure 3: *A: Lévy curve constructed by unfolding a Koch curve, and the example of a new generalization associated to higher-order harmonic trees $H(N, b > 2)$. B: schematic diagram showing the branch planes and the axes of rotation. C: image sequences of the unfolding process applied to $H(6, 2)$ (left) and $H(6, 3)$ (right).*

generalized to higher-order harmonic trees $H(N, b > 2)$, where each group of b branches is rotated from its common node around the same axis encountered in the binary case. The right column in Figure 3 C shows the unfolding process of the ternary harmonic tree $H(6, 3)$. The fractal curves associated to higher-order harmonic trees introduce a different Lévy-Koch generalization from the one encountered in the literature [6]. It is also important to note that the tip set of these trees remains connected and it has a constant fractal dimension regardless of the angle β. If we remove all the branches except the tip set we obtain a closed 3D-dimensional curve of infinite length and nowhere differentiable.

Mapping the Harmonics in the Forest of Symmetric Fractal Trees

Once we have the initiator (the trunk placed at diagonal d_b) and the generator (the first left branch placed at the polygon's left side), the harmonic tree is ready to be reconstructed following three simple steps: add the remaining $b-1$ branches separated by an angle of $\frac{2\pi}{N}$ radians to each other, generate recursively the fractal tree and ensemble $N-1$ copies of itself around the trunk's base. Consequently, if we set the trunk to be of length 1 we can map all the harmonic trees by the first left branches endpoints, see Figure 4. The dashed straight lines link together the $N-1$ harmonic points defined by the same N-gon. Harmonic binary fractal

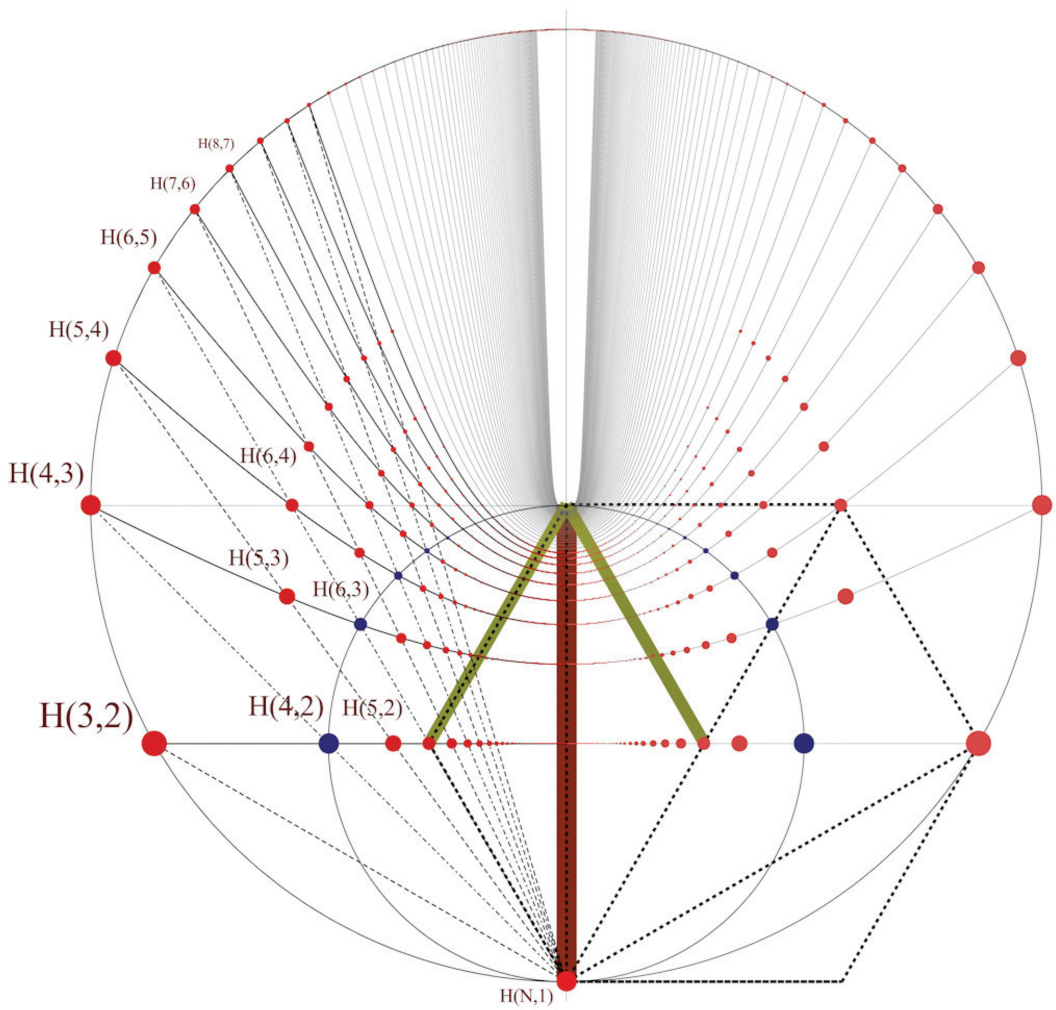

Figure 4: *The maps of the harmonics in the forest of symmetric fractal trees. Notice that we have superimposed the generator of $H(6,2)$ in the diagram to help the reader understand how the information contained in each point translates to the geometrical approach discussed earlier.*

trees are given by the points placed along the lower horizontal line. Harmonic ternary points are placed along the upper next curve, quaternary points on to the next one, and so on. The parametric equations of these curves are given by $r = -csc((\pi - b\theta)/(b-1))sin((\pi - \theta)/(b-1))$ where θ is the angle between the positive vertical axis and the first left branch, see Figure 6 to understand how these equations translate to the geometrical approach. When a point is reached, the scaling ratio of the corresponding harmonic tree $H(N, b)$ has the value $r = csc(b\pi/N)sin(\pi/N)$ and the first left branch is placed at $\theta = (1 + N - b)\pi/N$. The points contained in the diagram's half-unit disk and the ones on its boundary, $H(2b, b)$, represent harmonic fractal trees that have tip-to-tip self-contact, their tip sets are connected but they don't overlap, having thus a neat fractal dimension associated with: $D_{H(N,b)} = log(b)/log(1/(csc(b\pi/N)sin(\pi/N)))$ which ranges from dimension 2 for the $H(4, 2)$ tree, to dimension 1 for the limiting trees $H(\infty, b)$. The first six binary and ternary harmonic fractal trees are listed in Figure 5 and they are accompanied by their dual forms, $\beta = 180°$.

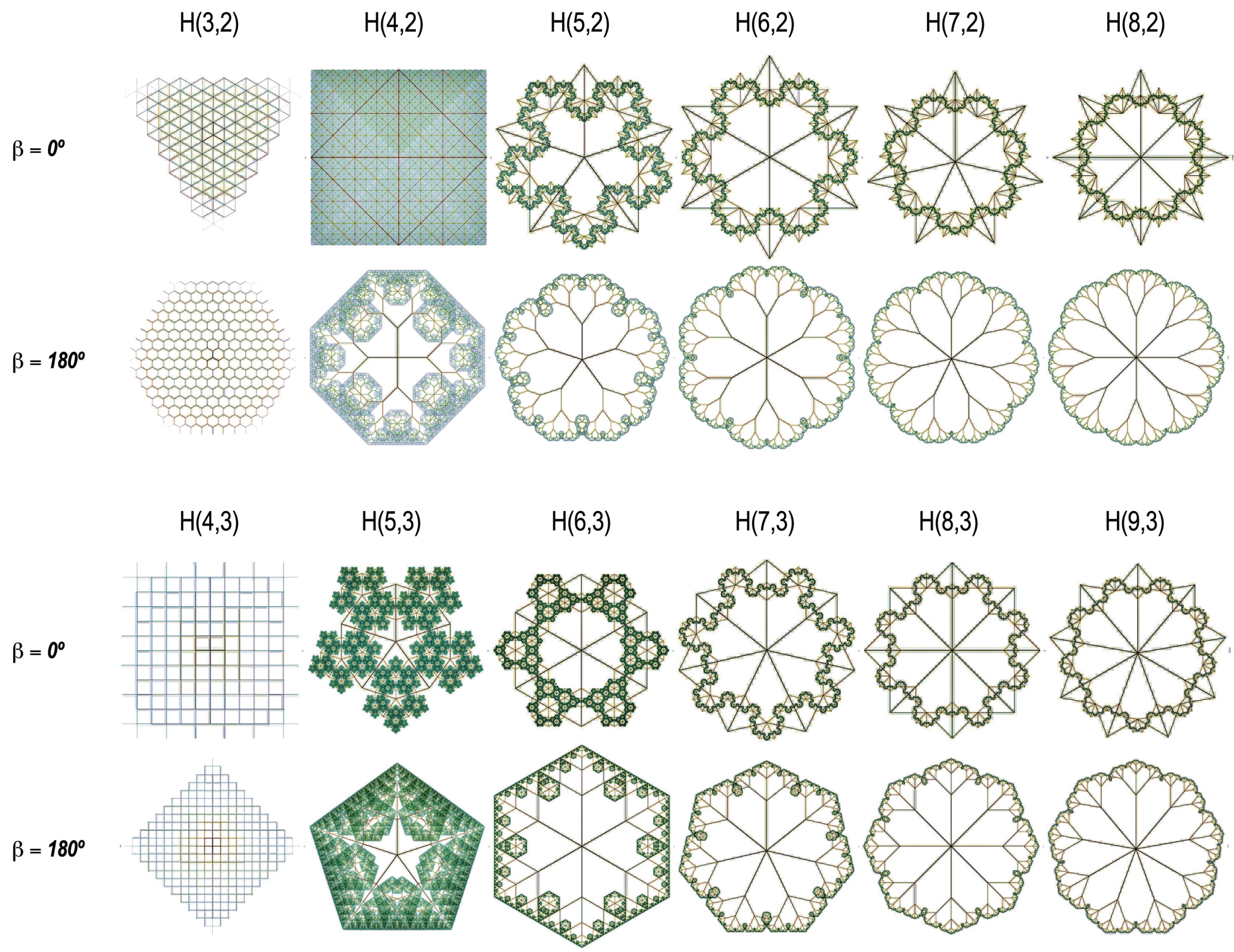

Figure 5 : *The first six binary and ternary harmonic trees, $\beta = 0°$, accompanied by their duals, $\beta = 180°$.*

Fathauer's Fractal Compendiums [2] [3] incorporate interesting fractal knots and fractal tilings that have among others the unfolded duals of $H(6, 2)$, $H(6, 3)$ in their underlying structure. In the 2000 Bridges Conference, Fathauer [4] presented two infinite families of *Lévy-like* fractal tilings associated to triangles and trapezoids of regular polygons. He suggested that they could be generalized to segments of higher number of polygon sides as it has been done in the present work. Finally, Tara Taylor presented the golden binary harmonic tree $H(5, 2)$ in the 2007 Bridges Conference [9] pioneering thus in pointing out the N-fold symmetry of certain self-contacting symmetric binary fractal trees. Here we have also presented the dual form of $H(5,2)$, the golden ternary harmonic tree $H(5,3)$ and its dual form, see Figure 5. Another result added by the present generalization is the unknown harmonic fractal tree $H(8,3)$ which exactly fits in the self-similar design of Castel del Monte, see for example Susanne Krömker's reconstruction using the underlying structure of the $H(8,3)$ dual [5].

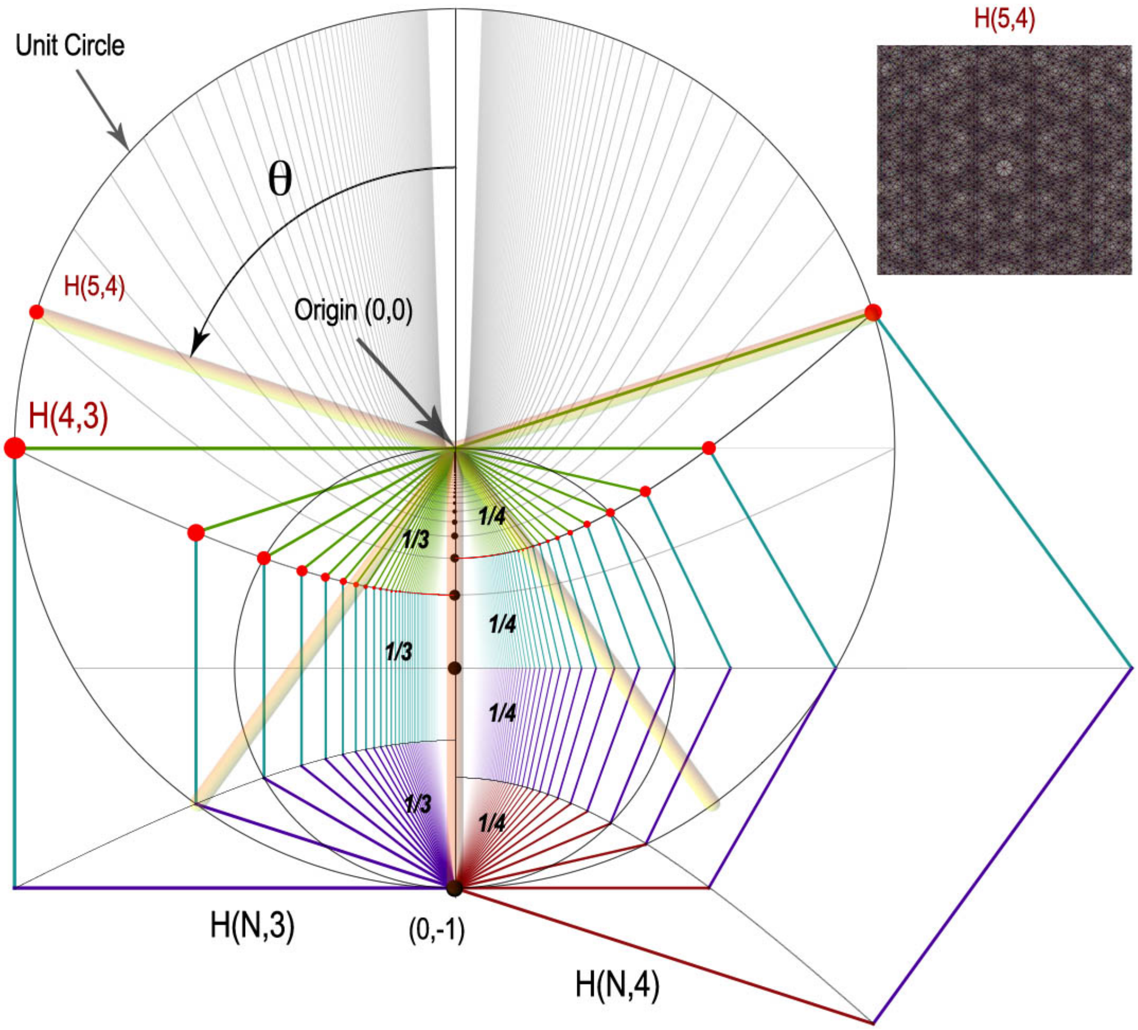

Figure 6: *The diagram map superimposed to the geometrical approach of $H(N,3)$ (left) and $H(N,4)$ (right). The black dots placed vertically represent harmonic trees with branch ratios equal to the harmonics of a vibrating string $1/2$, $1/3$, $1/4$, ... They are trees of the form $H(\infty,b)$ and their ratio is given by $r = 1/b$. The geometrical approach gives a proof by picture of the ratios $r = 1/3$ and $r = 1/4$ of the limiting elements $H(\infty,3)$ and $H(\infty,4)$ respectively. The tip set of these trees form a perfect circle which has an exact dimension 1.*

Platonic Solids and Harmonic Fractal Trees

Each point in the diagram's unit circle corresponds to a harmonic tree of the form $H(N,N-1)$, i.e. trees with $N-1$ branches per node and $r = 1$. These trees are particularly interesting because they end up filling the entire plane with an N-fold rotational symmetry lattice, and some of them suggest the possibility to divide the entire 3D space when unfolded by certain angles β. Figure 7 shows what happens for the $H(3,2)$, $H(4,3)$ and $H(5,4)$ harmonic trees when they are unfolded by and angle β equal to the dihedral angles of the Platonic Solids. The generalized Lévy-Koch curves introduced earlier produce the five platonic solids when the angle β reaches the dihedral angles: $\beta = arccos(1/3)$ for the tetrahedron, $\beta = arccos(-1/3)$ for the octahedron, $\beta = arccos(\sqrt{5}/3)$ for the icosahedron, $\beta = 90°$ for the hexahedron (cube) and $\beta = 2arctan((1+\sqrt{5})/2)$ for the dodecahedron. The 3D-dimensional space lattices correspond from left to right, to the β-folded $H(3,2)$, $H(4,3)$ and $H(5,4)$ harmonic trees.

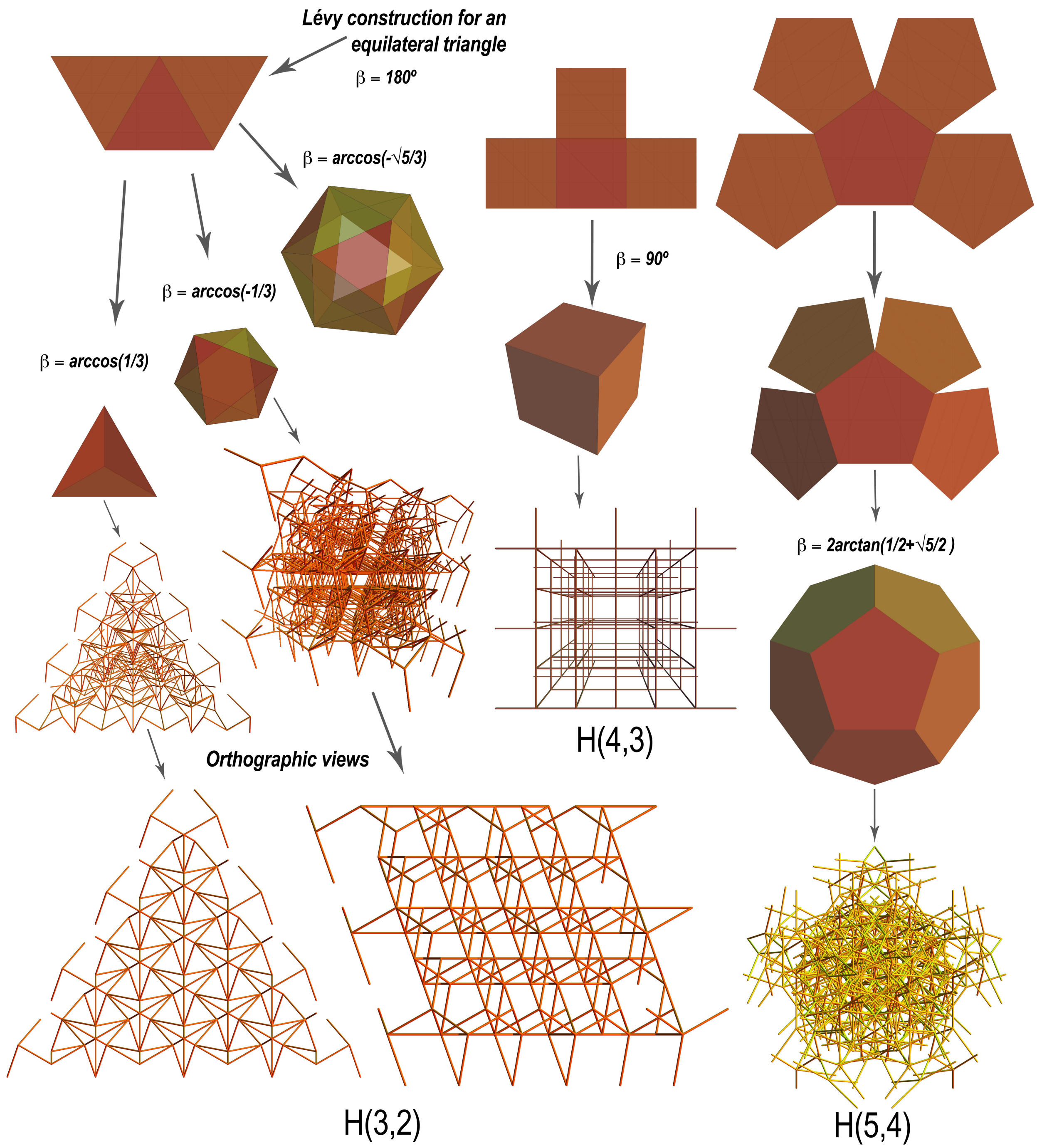

Figure 7 : *Recursively generated Platonic Solids and their associated harmonic fractal trees lattices.*

Final Remarks

Harmonic fractal trees emerged when the author was studying a broader class of symmetric fractal trees that satisfy tip-to-tip self-contact. The half-unit circle in the diagrams of Figures 4 and 6 contains the first order topological critical points for self-contacting symmetric fractal trees. From the previous works of Mandelbrot, Frame and Taylor on the special case of self-contacting symmetric binary fractal trees [8][9], the author has generalized them to any number of branches per node. His work can be found in a dedicated website to fractal trees [1], with fully detailed maps, galleries, animations and interactive *Mathematica's CDF documents*, as well as an extensive annotated bibliography. Everyone is invited to explore this mathematical forest of beauty and harmony.

Acknowledgments

The author thanks Susanne Krömker for her enthusiastic support of the author's research during an Erasmus exchange at Heidelberg Universität. Susanne's past research on fractals at her *Visualization and Numerical Geometry Group* inspired the author in many different ways. The initial research for this paper was partially done under the inestimable help of Jofre Espigulé, author's twin brother, and it was enriched with many constructive comments of several professors and friends including Warren Dicks, Gaspar Orriols and Carlos Castro.

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