The 6-ring

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Abstract

The 6-ring is a tubular surface of genus 13, obtained by gluing together twenty-four 12-gons which follows the regularity of the map $R13.2'\{12,3\}$. It is constructed from six rings of two Borromean rings, has the twenty-four elements of an oriented cube and matches nicely with the 6-coloring of $R13.2'\{12,3\}$. The 6-ring has minimal twists and geometrically equivalent sets of four 12-gons. We explicitly construct this highly symmetric surface from two Borromean rings together with a detailed mapping of the 12-gons.
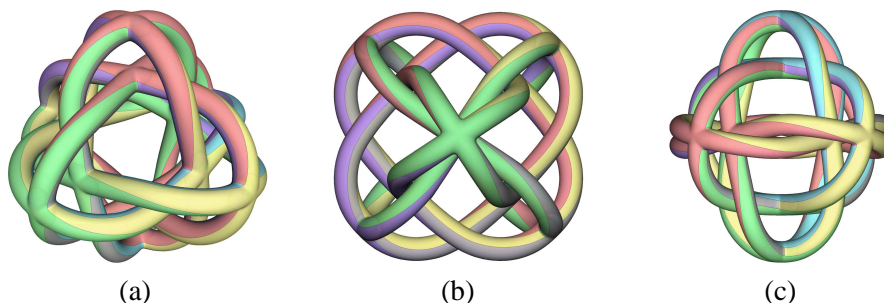
(a)　　　　　　　　(b)　　　　　　　　(c)

Figure 1 : *The 6-ring — a 3D realization of the regular map $R13.2'\{12,3\}$.*

Introduction

Suppose we have twenty-four 12-gons and we want to glue them at their edges so that any three of them share a vertex. There are several ways to solve such a problem, but we are interested in finding the most symmetric and the most visually appealing solution. Stated more generally: Given the triple (F,p,q), visualize a compact surface with F faces such that each face has p edges and any q faces share a common vertex.

Consider a 3D model of the cube. Explicitly count the number of its faces and edges (see Figure 2 (a)). So the cube is a solution to the triple $(6,4,3)$. In our problem we begin with a triple, for example, $(4,4,4)$ and we must find a corresponding 3D model. One solution would be to start with four squares in the plane then glue them around a vertex with the correct identification at the boundary. This is a topological torus which is a *non-compact* solution of our puzzle (Figure 2 (b)). A 3D compact model is obtained by using the parametric equation of the torus (Figure 2 (c)). We could extend this idea to solve the general problem, but it is difficult, if not impossible, to have a closed parametric equation of such a complex surface. Nevertheless, for any (F,p,q), a non-compact solution exists. We give an example of a solution of the triple $(16,3,8)$ in Figure 2 (d), where the identifications at the boundary have to be carefully chosen in order to make the surface topologically (but not geometrically) closed and orientable. In this paper, we give a compact solution of $(24,12,3)$ with maximal symmetry. The mathematical statement of this problem is to find a 3D embedding of the regular map $R13.2'\{12,3\}$.

A regular map is a family of equivalent polygons glued together to form a compact 2-manifold which is topologically vertex-, edge- and face-transitive. Some well known regular maps are the Platonic solids. The *i*-th reflexible regular map of genus *g* is denoted by $Rg.i\{p,q\}$, where p and q are the same as in our puzzle. The

dual regular map is denoted by R$g.i'\{q,p\}$. There are, for example, 12 regular maps (including duals) of genus 2, and the number increases for higher genus; for genus 13 there are 44. Visualizing such highly symmetric surface is challenging but by using the Borromean rings, we propose a model of R13.2$'$ having the twenty-four elements of an oriented cube.
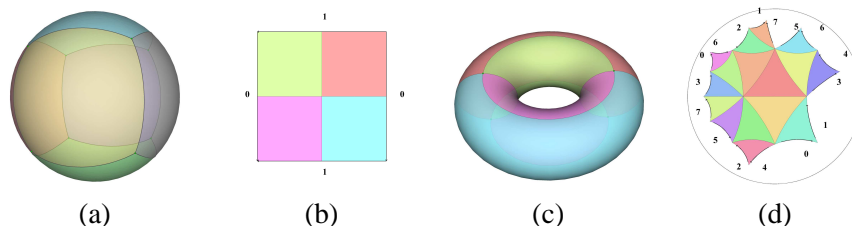

| (a) | (b) | (c) | (d) |

Figure 2 : *(a) the Cube* $(6,4,3)$*, (b) a non-compact torus, (c) a compact torus and (d) a non-compact solution of* $(16,3,8)$*.*

The Borromean rings, named after the Borromeo family in Italy, are three rings link together such that removing one of the ring leaves the other two unlinked. They are a simple example of Brunnian links. The Borromean rings have been used in different contexts for many centuries as a sign of strength through unity. They already appeared in 7 th century in Stona Hammers as the Valknut (a symbol consisting of three interlocked triangles, see Figure 3). They have been, for example, used as a symbol of the Christian Trinity and as a label of the Ballantine's beer. In 2006, a tight version of the Borromean rings has been adopted by the International Mathematical Union for its new logo. John Sullivan [18] said that *It represents the interconnectedness not only of the various fields of mathematics, but also of the mathematical community around the world.*
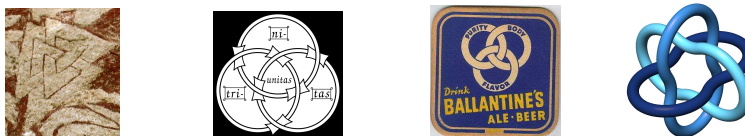

Figure 3 : *Examples of Borromean rings (from left to right): the Valknut in Stora Hammars, the Christian Trinity, Ballantine's beer label (image source: wikipedia [19]) and the IMU logo.*

Related works and contributions

Related works: Coxeter and Moser [3] is a nice introductory book about symmetry groups and regular maps. The visualization of regular maps is still an active research subject. The first attempt goes back to Carlo Séquin [15] inspired by Helaman Ferguson's sculpture *"The Eightfold Way"*, presented in Levy [7]. Séquin's investigations ([15], [16], [17]) are a huge source of inspiration to start modelling regular maps. He uses all possible modelling techniques, including sketches, paper models and Styrofoam, to finally obtain a computer generated model. He succeeded in solving several cases from genus 2 to genus 5 but each regular map is handled separately. An automatic algorithm was desired. Jack van Wijk [8] suggested a heuristic procedure to visualize some of the regular maps. His technique combines computational group theory, hyperbolic geometry and computer graphics. His method succeeded in solving several cases, but was too restrictive to handle even some of the genus 2 regular maps. In 2010, Séquin [17] filled the gaps missing in the genus 2 and genus 5 regular maps. To the best of our knowledge, no new models have been published since then. In our previous work [11], we suggest an energy based relaxation scheme to improve the shape of van Wijk's candidate regular maps. This energy does not only improve the symmetry of the shape but also optimizes the faces of the regular map to be as geometrically equivalent as possible.

Contributions: We take one of van Wijk's solution of R13.2' and show that by using our spring energy based relaxation scheme with topological constraints, we obtain the 6-ring (Figure 1) as a minimizer. We show that the 6-ring can be constructed geometrically from two Borromean rings and we give a simple visual process (without using any group theoretical result as in [8]) of gluing the twenty-four 12-gons on the resulting surface. Finally, we show that by choosing a 6-coloring of R13.2', we can choose a set of four 3D 12-gons as a fundamental domain.

Construction of a genus 13 surface

In this section, we present several ways of constructing a genus 13 surface and describe their properties so that we can decide which of them work best as a space model of R13.2'. But first, we review some basic results from surface topology. The genus g of a surface is the number of "tunnels" in that surface. This can be computed by the Euler characteristic, which for a compact surface is defined by $\chi = 2 - 2g$. The constant χ is obtained by the relation $\chi = V - E + F$, where V is the number of vertices, E the number of edges and F the number of faces of the surface. The Euler characteristic is a constant that describes the shape of a topological space independent of its geometric properties. It can be used, for example, as a necessary and sufficient condition for two compact surfaces to be homeomorphic (a continuous deformation of one shape to another, for example, a tea pot and a donut). For a low genus tubular surface, it is easy to determine its genus, by counting the number of tunnels in the surface. For high genus tubular surfaces, we use the following method to find their genus. We partition each tube into two half tubes—so we have 2 edges and 2 faces—and each junction into 2 vertices and α_j edges, where α_j is the number of tubes meeting at the junction j. Using the Euler characteristic, we can derive the following relation: $g = 1/2 \sum_{j=1}^{N_{\text{junc}}} \alpha_j - N_{\text{junc}} + 1$, where N_{junc} is the number of junctions of the surface. This is equivalent to the first Betti number or the cyclomatic number, $g = E - V + 1$ by considering each tube as an edge and each junction as a vertex.

To generate high genus surfaces, a common technique is to use tubular structures obtained from the edge graph of already existing maps. In general, the map does not need to be regular and we should not restrict ourselves to tubular surfaces; artificial high genus surface can also be generated as in Figure 4 (c). van Wijk suggested a selection criteria in order to map a given tiling on high genus surfaces, which we adopt here. This can be summarized by two fundamental conditions: (i) either the p-gons are centered at the junctions and the number of tubes meeting at the junctions is a divisor of p or (ii) some vertices of the p-gons are placed at the junctions and the number of tubes meeting at the junctions divides q. Even though these conditions are too restrictive and even fail to solve many cases, they are natural steps to glue tiles on tubular surfaces. For more details we refer to [8].

Let us now give several genus 13 surfaces and see which one of them agree with (i), (ii) and R13.2'. Our first candidate is the most symmetrical genus 13 surface. It is obtained from a truncated cube called a *CubeOctahedron* which inherits the full symmetry of the cube, see Figure 4 (a). We call it Tub(CubeOcta). Our second candidate has less symmetry, obtained from a hosohedron $\{2, 14\}$ and has only one C13-axis, see Figure 4 (b). We call it Tub(Hoso-14). And our last candidate is a manually designed surface which also has a C13-axis but with a different topology at the junctions, see Figure 4 (c). We call it M_{13}. All these surfaces have genus 13 but with different junctions.

Condition (i) already fails for all three of them since Tub(CubeOcta) has ten junctions, Tube(Hoso-14) has only two, and M_{13} has 14, none of which has the desired 12 (each junction should contain two 12-gons of R13.2'). Condition (ii) does not work either since 4 does not divide 3 (for the case of Tub(CubeOcta)) and 14 does not divide 3 (for the case of Tube(Hoso-14)). M_{13} has one junction consisting of 13 tubes and 13 junctions consisting of three tubes. They all divide three but the fact that it has different numbers of tubes at the junctions makes it hard to imagine a natural process of gluing the 12-gons on it. Hence, we exclude it
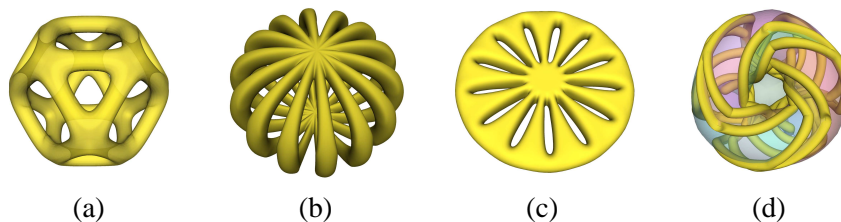
from our search space.

(a)   (b)   (c)   (d)

Figure 4 : *Examples of genus 13 surfaces obtained from a: (a) CubeOctahedron, (b) hosohedron {2,14}, (c) manual design and (d) R3.4'{6,4}.*

What we are looking for is a tubular surface having exactly 12 junctions and the number of incoming tubes at each junction must be a divisor of 12. This is equivalent to finding a surface with 12 vertices and some number of faces such that 2,3,4,6 or 12 of them share a vertex. This space is huge but by restricting ourselves to p-gon faces, we can find a regular map with these properties. R3.4'{6,4} is an example of one: it has 12 vertices and eight hexagons such that four of them always share a vertex. A space model of it is depicted in Figure 14 of [8]. This model is obtained from a hosohedron {2,4} but we will use a model obtained from a tetrahedron (see Figure 4 (d)) as a target model. By tubifying the edge graph of this map, we obtain the desired genus 13 surface, which we call Tub(R3.4'). Our case analysis is an alternative view of van Wijk's method of generating regular maps. As we can see from Figure 4, although the surface suggested in (d) fulfill all the conditions, it lacks symmetry and elegance. This is where van Wijk's method fails. In our earlier work [11], instead of using a direct tubification of the edge graph, we applied a spring relaxation energy and let the tube deform accordingly. We will see in the next paragraph that the tube frame of the 6-ring is a minimizer of Tub(R3.4').
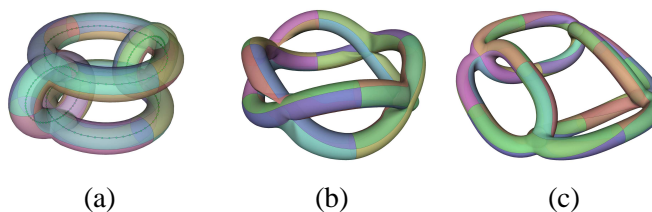

(a)   (b)   (c)

Figure 5 : *(a) R5.3'{5,4} and its medial axis, (b) relaxation without constraints, (c) relaxation with constraints.*

Energy based modelling: The idea is to consider each vertex as a charge particle attracted by its neighbors and repulsed by all other vertices. The total energy of this system is given by

$$E(G) = \left\| \sum_{i=1}^{|v|} \sum_{i \neq j} f_r(x_i, x_j) + \sum_{i=1}^{|v|} \sum_{i \leftrightarrow j} f_a(x_j, x_i) \right\|^2$$

where, $f_a(x, x_j) = H \|x - x_j\|(x - x_j)$ is the force of attraction exerted by x_j on x and $f_r(x, x_j) = K \frac{1}{\|x - x_j\|^2}(x_j - x)$ is the force of repulsion. H and K are constants defining the strength of the forces. This energy is used by Scharein [12] for relaxing knots in KnotPlot, with a knot type preservation constraint. This energy is also used in the smoothing of Seifert surfaces used by van Wijk and Cohen in [10]. For our purpose, this energy alone is not enough to guarantee a non-self-intersecting surface. We need an extra topological constraint at the nodes of the graph. Suppose that x is a node vertex and link$(x) = \{x_1, \ldots, x_n\}$ is the set of all vertices adjacent

to x. Then, for a vertex $x_i \in \text{link}(v)$, we add a planarity constraint on x_i such that for all $x_j \in \text{link}(v)$, with $j \neq i$, x_j exerts a force of magnitude $L(E\sqrt{\sin^2(2\pi i/n) - \cos^2(2\pi i/n)} - d_{ij})$ on x_i, where d_{ij} is the euclidean distance between x_i and x_j, L is a constant and E is the average edge length of the edge graph. There are still some extra forces that we added to the system to have more control of the final shape, more details can be found in [11]. In Figure 5, we took the regular map R5.3'{5,4} and showed that without the topological constraints at the node, the resulting minimized shape has self-intersections. This modified spring energy works well and improves the symmetry of the regular maps in most cases. The tubular shape of the 6-ring is a minimizer of $E(G)$ applied to the edge graph of the model obtained in Figure 4 (d). The minimization process is illustrated in Figure 6. Unfortunately, it is not unique. Luckily the system did not get stuck at a local minimum. This happens often in our experimentation with other shapes. Inspired by the symmetry of the 6-ring, we were convinced that a more precise geometrical approach would be more convenient. We are going to use two Borromean rings to achieve that.
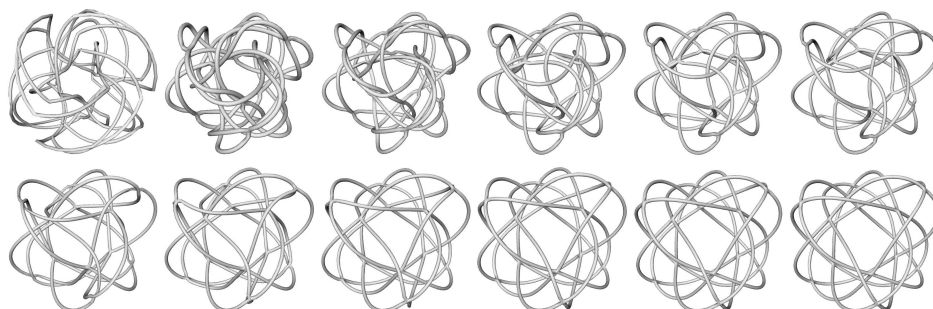

Figure 6: *Spring energy minimization applied to the medial axis of Tub(R3.4') to obtain the shape of the 6-ring.*

Using two Borromean rings: A way to generate the Borromean rings is to start with an ellipse with the following parametric equation:

$$(x(u), y(u), z(u)) = (\cos u, 0, 2\sin u) \quad 0 \leq u \leq 2\pi \tag{1}$$

and perform successive 90 degree rotations around its canonical axis to obtain the other ellipses. This construction is shown in Figure 7. We then take the final ring, duplicate it and rotate the duplicated links around the z-axis as shown in Figure 7 (d). The resulting tubular surface that has the links as its medial axis, has genus $g = 21$. To derive a genus 13 surface, we remove eight of the junctions by not letting the two Bor-
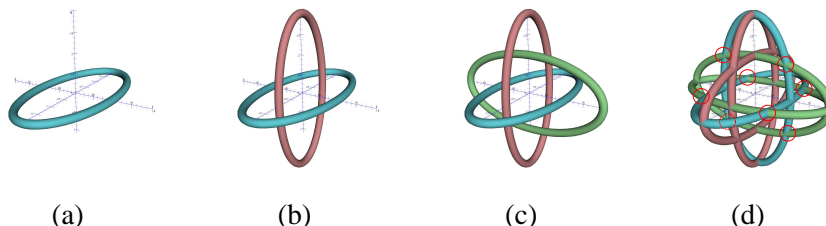

(a)　　　　　　(b)　　　　　　(c)　　　　　　(d)

Figure 7: *Construction of the Borromean rings from an ellipse and a genus* 21 *surface obtained from two Borromean rings.*

romean rings intersect at some points; these intersections are marked with circles in Figure 7 (d). As a result, we have only 12 junctions and we obtain our genus 13 surface. The idea is to start with a non-planar ellipse which looks like a *figure-8* in the xy-plane. Such ellipses can be parametrized by

$$(x(u), y(u), z(u)) = (a\cos u, b\sin(2u), c\sin u), \quad 0 \leq u \leq 2\pi, \tag{2}$$

where $a > 0, b \in \mathbb{R}$ and $c > 0$. a and c are the radii of the ellipse and b controls the flatness of the *figure-8* in the *xy*-plane. In our model, we chose $a = 1, b = -1/5$ and $c = 2$. We can see from Figure 8 (c) that there are exactly 12 intersection points produced by the two Borromean rings. To obtain the final surface, we use these Borromean rings as a medial axis and build a tube around it.
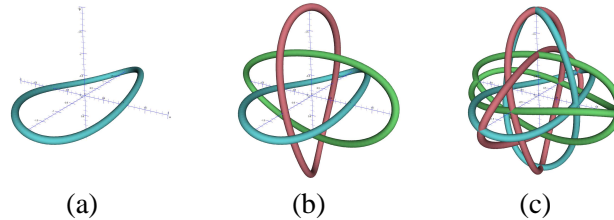

(a) (b) (c)

Figure 8 : *The Borromean rings with non planar ellipses and the medial axis of the 6-ring.*

Intuitive gluing of the 12-gons

In this section, we recall how to generate a non-compact representation of R13.2' in hyperbolic space as illustrated in Figure 2 (d). It is not possible to tessellate the sphere with regular 12-gons, nor the Euclidean plane since the angle at the vertices does not sum up to 2π. However, it is possible in the hyperbolic space. Beardon [1] is a good introductory book for hyperbolic geometry. The usual way of generating a $\{p, q\}$ tiling of hyperbolic space is presented in Dunham [4]. His article provides a comprehensive way (with pseudo code) to generate hyperbolic tilings. The algorithm, unfortunately, only produces a universal covering (several copies of the regular map), and finding one regular map in such an infinite group is quite hard. A simple method to generate regular maps is to use their symmetry group. It is usually denoted as a set $\langle G | \mathscr{R} \rangle$ of generators G and relators $\mathscr{R}$. The symmetry group of all the regular maps of genus less than 302 have been enumerated by Conder in [2] in the form of generators and relators. R13.2$'$ has the following symmetry group,

$$\mathrm{Sym}(R13.2') = \left\langle R, S, T \,|\, T^2, S^{-3}, (SR)^2, (RT)^2, RSR^{-2}SR^{-1}SR^{-2}SR^2, R^{12} \right\rangle.$$

Here, R, S are rotations and T is a reflection. For a geometric definition of these isometries, see for example [11]. Sym(R13.2') is a finite group and its element representatives can be enumerated using Advanced Coset Enumerator (ACE) [6]. ACE gives the coset table of a given finitely generated group. Using ACE, we found that Sym(R13.2') has order 576. It can be realized as a triangular tiling group in the hyperbolic space obtained by a fundamental triangle which has angles $\pi/p, \pi/q$ and $\pi/2$ (see Figure 9). This triangle group can be partitioned into twenty-four 12-gons by taking the quotient of the subgroup $H_1 = \langle R, T \rangle$ in Sym(R13.2'). According to Senechal [14], this induces a 24 symmetric colorings of the tiling group. We can assign a unique color to every four of these twenty-four 12-gons, by taking the quotient of the subgroup $H_2 = \langle R, T, SR^{-3}S \rangle$ in Sym(R13.2'), which induces a 6-coloring of the map. This coloring technique is called *left coset algorithm* and is used to color hyperbolic patterns (see for example Penas [5]). A nice feature of this method is the possibility to extend the color symmetry to the universal covering of the map. The symmetry group of the regular map not only gives us a way to generate the 12-gons but also provides us with information on how these 12-gons should be glued together so that the color permutation at each vertex is preserved. We choose the 6-coloring of the map since it matches nicely with the symmetry of the 6-ring.

Gluing the 12-gons: Once the neigbhoring information of the 12-gons are obtained in the hyperbolic space, we can start to glue them on to the tubular surface. We will only glue the four red tiles (see Figure 9 (c)) and generate the rest by applying rotations on these first four. Here a precise geometric construction of
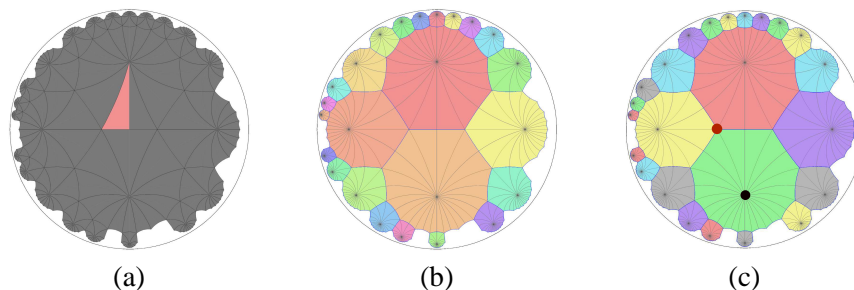
Figure 9 : *(a) A fundamental triangle having angle $\pi/2, \pi/p$ and π/q, (b) a twenty-four 12-gons partition of the triangular group and (b) the 6-coloring of R13.2'.*

the target surface is needed since the rotations should map edges to edges and cover the whole surface. For each 3D rotation, we identify the corresponding rotation in the hyperbolic space. From here, the 12-gons are considered to be an elastic fabric and are allowed to be stretched and twisted during the gluing process. We place the center of the tiles at the junctions and the vertices are placed where two tubes meet at the junctions. This process is summarized in Figure 10. To obtain the other 12-gons, we use the C4-axis of the surface as in Figure 1 (b) to get the purple, grey and yellow 12-gons. This C4-axis is equivalent, in hyperbolic space, to a clockwise rotation around the center of a green 12-gon (black dot in Figure 9 (c)) with an angle $2\pi/p$. We use the C3-axis in Figure 1 (a) to get the green 12-gons, which is equivalent, in hyperbolic space, to a clockwise rotation around the vertex where a red, yellow and green 12-gons meet (red dot in Figure 9 (c)) with an angle $2\pi/q$. All the remaining symmetry axes of the 6-ring can be identified in the same fashion as a rotation in the hyperbolic space. A combination of these two rotations gives the blue 12-gons and finally we obtain a model of R13.2' with, first, the same color permutation in the hyperbolic space and second, the same number of symmetry elements which is in this case twenty-four.
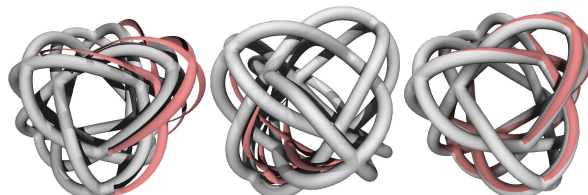

Figure 10 : *Gluing of the four red 12-gons on the surface forming a fundamental domain of the 6-ring.*

Conclusion

We presented a mathematical construction of the 6-ring, a highly symmetric surface formed by twenty-four 12-gons. These 12-gons are glued together with a 6-coloring scheme according to the symmetry group of R13.2′. The resulting faces of the map are not only topologically equivalent but by taking appropriate starting 12-gons, all the remaining 12-gons can be obtained by rotations around some specific axis. We modelled the Borromean rings with explicit formulae to derive the genus 13 of the 6-ring. The harmony of the 6-ring motivates us to further study and improve the symmetry of the existing regular maps in the hope of finding a more conceptual approach to tackle the general problem of visualizing them. There are embeddings of regular maps which have all *p*-gons that are geometrically equivalent. Examples of them are shown in Figure 13 of [8]. The 6-ring has four 12-gons out of twenty-four as its fundamental domain which is already good for a genus 13 regular map. We would like to know if this is the best one can do, in other words, having three or two or even one fundamental 12-gons on the embedding. A nice exercise would be to classify all the existing embeddings having all or almost all *p*-gons geometrically equivalent. It would also be interesting to

explore other kind of knots and links to see what kind of genus g surface we will obtain from them.

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