Cross-Caps – Boy Caps – Boy Cups

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Abstract

We present a constructive, pictorial introduction to low-genus, non-orientable surfaces such as Möbius bands, cross-caps, and Boy's surface. We explore the use of these compact surface elements as general building blocks to make topologically equivalent models of arbitrary complex surfaces immersed in 3D Euclidean space, such as Klein bottles and single-sided surfaces of higher genus. As a by-product we generate some models for geometrical sculptures and for other whimsical artifacts, such as self-intersecting hats and furniture with unusual shapes.

1. Introduction

Most people are intrigued when they first encounter a Möbius band, a cross-cap, Boy's surface, or a Klein bottle. But even for people with some advanced mathematical education, these low-genus single-sided surfaces hold some mysteries. It is not immediately clear what the genus is of these objects, or how many different colors are needed to color any arbitrary map on those surfaces so that no two adjacent countries employ the same color. It is even harder to figure out how many structurally different versions of each type of surface exist that cannot be transformed into one another with a regular homotopy, i.e., with a smooth deformation that does not create any tears, or creases with infinitely high curvature, but allows the surface to pass through itself [6][9]. Here we will discuss relationships between such low-genus, non-orientable surfaces, and describe transformations that will take one surface into another one. The goal is to give readers a deeper understanding of these topological entities and their geometrical instantiations, and to enhance readers' enjoyment when they will encounter these shapes again in a different context.

Figures 1 through 3 briefly introduce the cast of characters and establish the names used in this paper. As a point of contrast we first show a few two-sided, orientable surfaces. A disk (Fig.1a) is a two-sided surface with a single rim. This manifold is topologically equivalent to a punctured sphere (Fig.1b) and can be deformed into it with a regular homotopy. If we close the puncture by sewing another disk onto the rim of the opening, we obtain a regular sphere, i.e., a closed, compact surface with an Euler characteristic $\chi = 2$. As a reminder, the Euler characteristic, χ, can be calculated by embedding a mesh in that surface, counting the number of vertices (V), edges (E), and facets (F) of that mesh, and then subtracting the number of edges from the sum of its vertices and faces: $\chi = V - E + F$. If we puncture the sphere with a second hole, we obtain an open cylinder (Fig.1c) with two rims, and with $\chi = 0$. Such a ribbon may also be twisted (Fig.1d), but as long as the twist is an integer number of full turns, it falls into the same topological equivalence class.

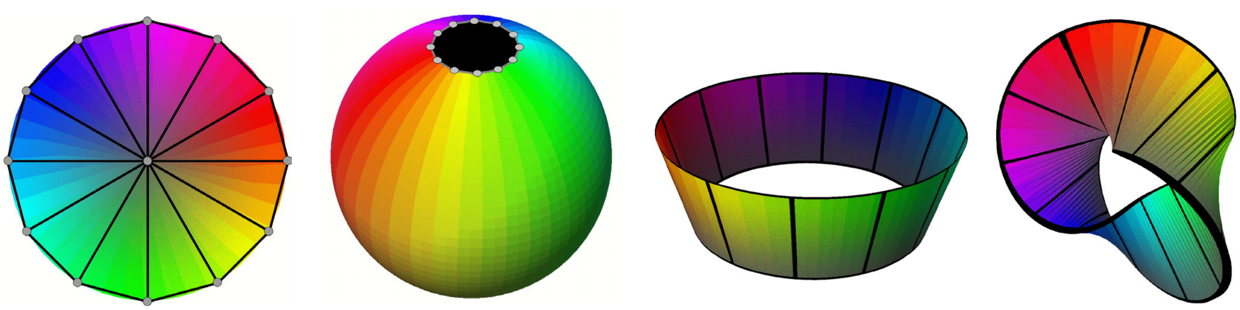
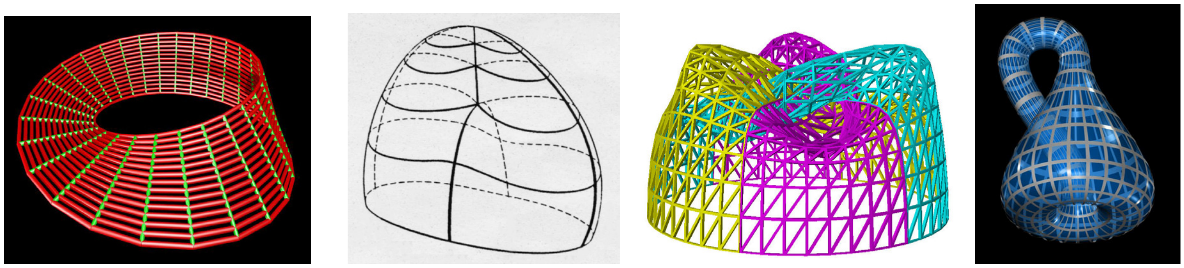
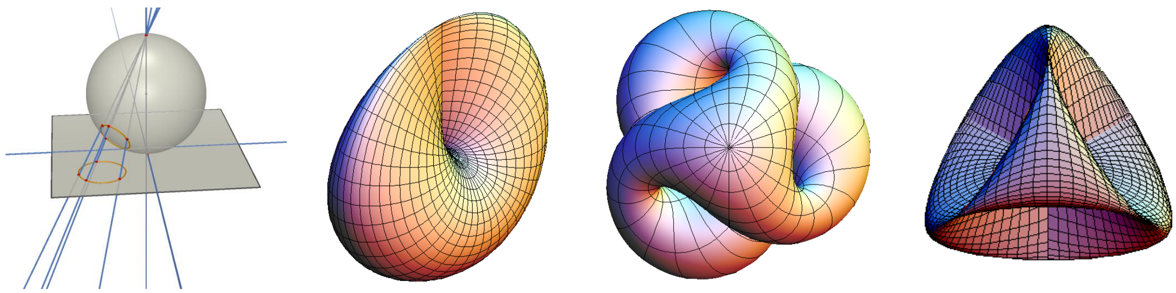

(a) (b) (c) (d)

Figure 1: *Orientable surfaces: (a) disk and (b) punctured sphere with χ=1; (c) open cylinder or annulus, and (d) a fully twisted ribbon, both with χ=0.*



(a) (b) (c) (d)

Figure 2: *Single-sided surfaces with χ=0: (a) Möbius band, (b) cross-cap, (c) Boy cap, (d) Klein bottle.*



(a) (b) (c) (d)

Figure 3: *Single-sided surfaces with χ=1: (a) projective plane, (b) closed cross surface, (c) left-twisting Boy surface (**BL**), (d) Steiner's Roman surface.*

Figure 2a shows a Möbius band – a non-orientable surface with $\chi = 0$, with a single ("double-loop") rim. This entity is topologically equivalent to the cross-cap (Fig.2b) and the Boy cap (Fig.2c), which both have a simpler, circular open rim (at the bottom), but at the expense of a more complex mapping into 3D space comprising some self-intersections. (Fig. 5 shows a transformation between 2a and 2c.) Figure 3 shows the conceptual generation of the projective plane (3a), and some compact models of it, obtained by closing off the open edges of the surfaces shown in Figures 2a-c by grafting disks onto their rims, as we did before to make a closed sphere. The result may be a cross surface (Fig.3b) or a Boy surface (Fig.3c), respectively. These closed surfaces with $\chi = 1$ cannot be shown in 3D Euclidean space without exhibiting some self-intersections. We can construct an even more symmetrical model of the projective plane: Steiner's *Roman surface* [17] has full tetrahedral symmetry, but at the cost of six singular points, forming so called *Whitney Umbrellas*. As an alternative, we may choose to close the rim of a Möbius band with another Möbius band; this will then yield a Klein bottle (Fig.2d), a closed, single-sided surface with $\chi = 0$.

In the following we will look at some of these equivalences more closely. We will use these geometries to make some funky artifacts, and, in some instances, I will point out the potential of these mathematical visualization models to serve as maquettes for intriguing geometrical sculptures.

2. Some Key Transformations

Closing a Möbius Strip

It was mentioned above that a compact model of the projective plane can be obtained if we seal off the rim of a Möbius strip with a topological disk. If we draw the Möbius strip in a traditional manner (Fig.2a), the closing disk will be quite contorted. A possible closing process is illustrated in Figure 4: The closing disk is grown in two separate pieces, starting from two opposite locations on the rim of the Möbius band (Fig.4a,b). Successively we add more surfaces strips that span consecutive segments of equal color on the Möbius rim (Fig.4b-d). In the end, the half-disks join up in an intersection line (red) with the original Möbius band (Fig.4e) – forming the self-intersection segment of an ordinary cross surface (Fig.3b) with two singular points of infinite curvature at either end of it.

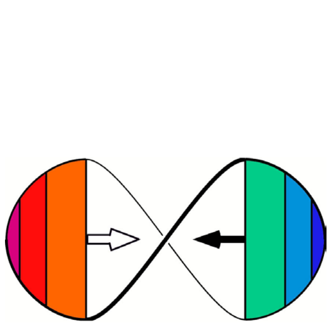


Figure 4: *Closing a Möbius band by growing two half-disks on its rim: (a) topological (top) view; side views: (b-d) first 3 pairs of closure segments added; (e) the finished result is a cross surface.*

For the other two variations of the Möbius band, the cross-cap and the Boy cap, we can just graft a disk or hemisphere to the bottom of these caps, since in both cases the surface emerges from that rim in a consistent (upward) direction; so the closing disk will not produce any additional intersections or singularities. This property also makes these caps (Fig.2b,c) useful as building blocks for constructing surfaces of higher genus. Whenever we start with some surface, punch a hole in it, and then seal off this hole with one of these caps, we obtain a non-orientable surface with its Euler characteristic, χ, lower by 1 than what we started from. For non-orientable surfaces the genus is defined as $g = 2–\chi$, i.e., equal to the number of cross-caps that we have grafted onto a sphere.

Transforming a Triply-Twisted Möbius Band into a Boy Cap

Since the Möbius band, the cross-cap, and the Boy cap are all topologically equivalent, we can try to transform these surfaces smoothly into one another. Such a transformation is depicted in Figure 5, and a short movie of this transformation can be seen on the web [7]: A triply twisted Möbius band (Fig.5a) is embedded in the surface of a Boy cap. Its rim is gradually extruded outwards, and in this process it forms the central triple point (Fig.5b), unwinds the three loops around the three tunnels (Fig.5c-d), and assumes an ever more circular shape (Fig.5d-e). Eventually it shrinks to a loop around a puncture in a Boy surface, which is equivalent to a Boy cap (Fig.2c).

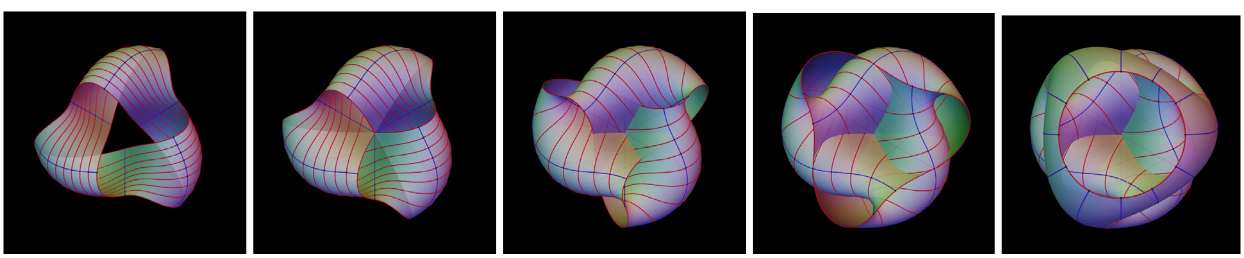
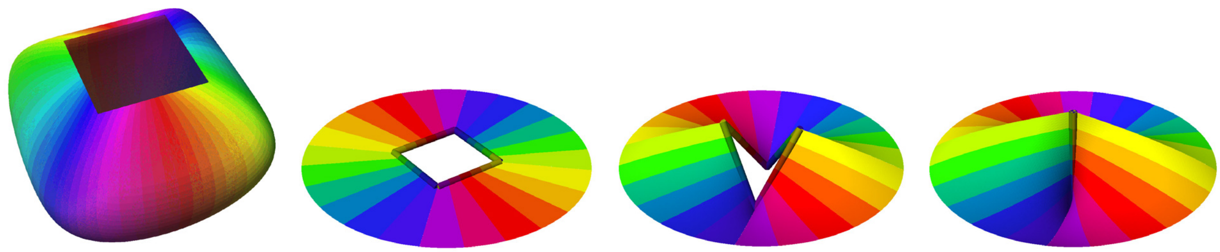


Figure 5: *Transforming a triply-twisted Möbius band (MR3=ML) into a punctured Boy surface [7].*

Forming a Compact Model of the Projective Plane with Cross-Cap Closure

Figure 6 gives a quick review of how to turn the infinite projective plane into a finite, closed model [11]. We first puncture the projective plane at infinity and shrink the remaining surface to where it lies entirely in the visible region. We reshape its border rim into a square frame and bulge out the surface within this quadrilateral frame into a large spherical sack (Fig.6a). The key task now is to close the square opening again in such a way that points lying at opposite positions (identically colored for easy visualization) join up – since this is the hallmark of the original projective plane: If we walk off to infinity in a particular direction, e.g. SW, we will return from the opposite direction, i.e. NE. This joining of opposite points can easily be accomplished by treating the square border as a linked loop of four equal length sticks (Fig.6b), which we then fold up in a zig-zag manner (Fig.6c). In the end, all four sticks will merge into a single line segment (Fig.6d) which forms the self-intersection crease of a cross-cap (Fig.2b) – or of a cross surface (Fig.3b) if we add back the bulging sack below this circular cross-cap. This surface has two singular points with infinite curvature, one each at either end of this self-intersection line segment.

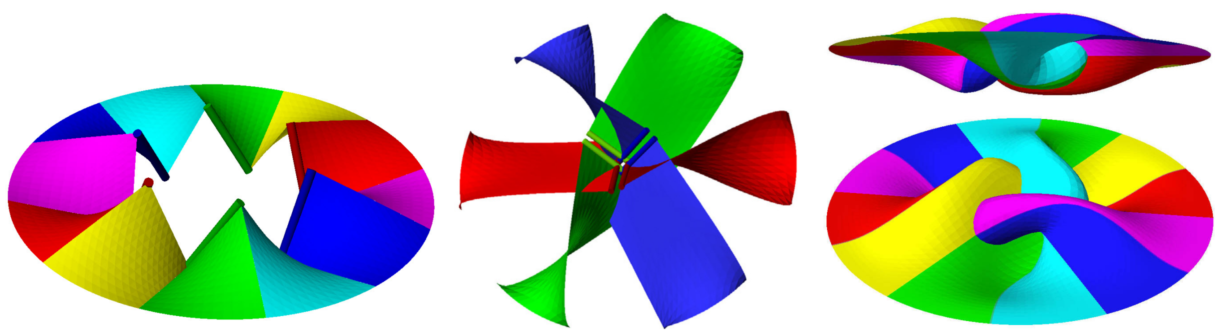


(a) (b) (c) (d)

Figure 6: *Modeling the projective plane: (a) a finite square domain bulging out beyond its frame, (b) 4-sticks frame in a flat annulus, (c) sticks in zig-zag formation, (d) final cross-cap.*

Smooth Closure with a Boy Cap

By starting from a 12-sided opening, we can obtain a smooth, single-sided closure by forming a Boy cap. Figure 7 illustrates the process. In this case we reinforce only every second side of the 12-gon with rigid sticks (Fig.7a) and treat the other 6 segments like rubber bands, which we will deform into loops later. Again we fold up this segmented contour, so that opposite sticks merge with reversed directions; but now we place the three line segments where those mergers occur at right angles to one another (Fig.7b). Thus in each pair one stick has to rotate through 54.7° and the other one through 125.3°. Opposite rubber bands also merge with the proper orientation, so that all opposite points in the rim will be joined. Now we shrink the lengths of the sticks and re-shape the rubber bands into ¾–circle loops to form three nicely rounded tunnels (Fig.7c). This is the defining geometric configuration of Boy's surface, the first and probably the simplest compact model of the projective plane that has no singular points with infinite curvature.



(a) (b) (c)

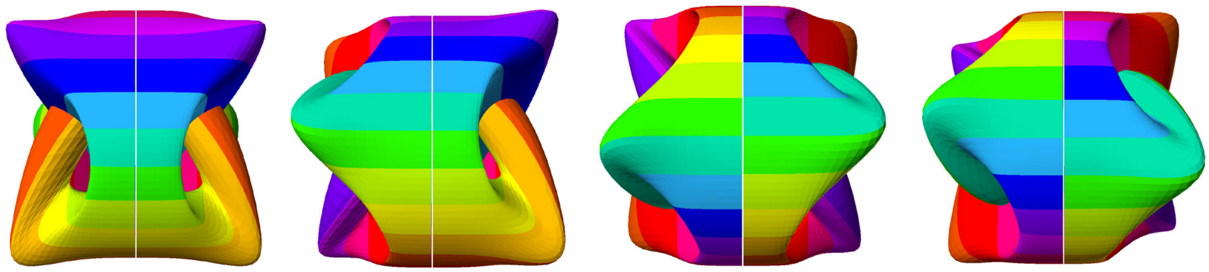
Figure 7: *Closure of a 12-gon hole into a Boy cap: (a) zig-zag rim, (b) the three sticks pairs almost merged into an orthogonal tri-hedron, (c) smooth surface closure (seen from side and from above).*

The same smooth closure can also be achieved with any larger odd number of opposing sticks pairs. If we start with an opening in the shape of a 20-gon and merge 5 pairs of opposite sticks, then we can construct a 5-tunnel Boy cap as used in Figure 12d. These smooth Boy caps can be used to form non-orientable surfaces of arbitrary genus by grafting the proper number of them onto holes punched into a sphere.

3. Single-Sided Surfaces with High Symmetry

It is well known that every Klein bottle can be described as a composite of two Möbius bands glued together along their edges [16]. Because of the topological equivalence of Möbius bands, cross-caps, and

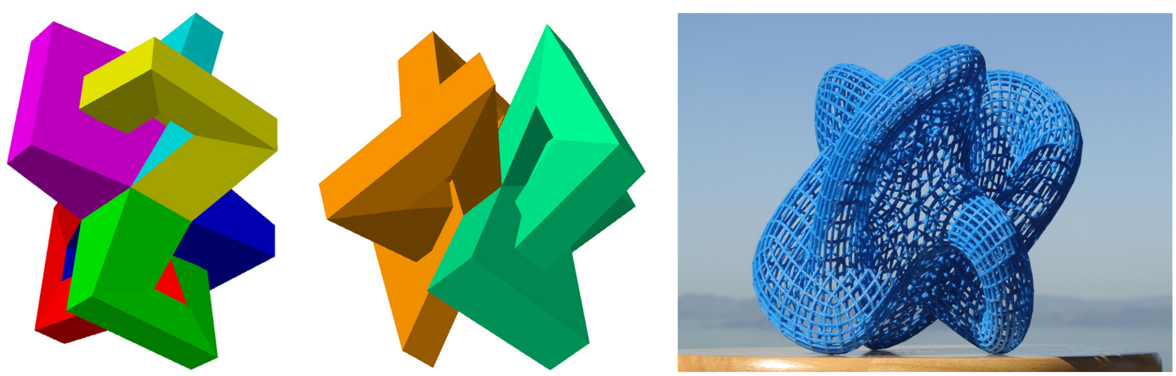
Boy caps, we can use any two of those elements to form a Klein bottle. In particular, by forming the connected sum of two Boy caps (**BL**, **BR**) of equal or opposite handedness we can construct all three types of Klein bottles with different regular homotopic structures: **BL** # **BR** = **KOJ**; **BL** # **BL** = **K8L**; **BR** # **BR** = **K8R** [16]. Using two 3-fold symmetric Boy caps of opposite chirality allows us to make Klein bottles with 3-fold symmetry as well as mirror or glide symmetry: Depending on whether we line up the three tunnel pairs or offset them by 60°, we obtain a shape with overall symmetry of type C_{3h} [10] {Conway notation: 3*} (Fig.8a) or S_6 {Conway: 3×} (Fig.8b). If we form the connected sum of two identical Boy caps, we obtain one of the Klein bottles with built-in chirality; those will always have D_3 symmetry {Conway: 223} regardless of the angle mismatch between the two halves (Fig.8c,d).



(a) (b) (c) (d)

Figure 8: *Boy caps of opposite chirality (a) joined with mirror symmetry and (b) with a 60° offset to exhibit glide symmetry; (c,d) two identical Boy caps always make a shape with D_3 symmetry.*

Attractive and instructive models can be built of these shapes by starting with a rough polyhedral outline of the desired combination of Boy caps (Fig.9a,b), then refining and smoothing that surface with a subdivision algorithm, and finally turning the facetted mesh into a grid structure. The resulting geometry can then be built with a rapid prototyping machine based on layered fabrication. The model exhibiting the case of S_6 symmetry (Fig.9c) was built on a fused deposition modeling (FDM) machine from Stratasys.



(a) (b) (c)

Figure 9: *Polyhedral models of the connected sum of two Boy caps: (a) **BL#BL**, (b) **BL#BR**; (c) gridded model with S_6 symmetry resulting from (b) realized on an FDM machine.*

Boy caps with 3-fold symmetry can readily be grafted onto the triangular faces of a Platonic solid to obtain a highly symmetrical single-sided surface of higher genus. Four identical Boy caps thus make a closed surface with the symmetry of the oriented tetrahedron (Fig.10a,b); eight Boy caps can yield chiral octahedral symmetry (Fig.10c), and 20 Boy caps can make an icosahedral structure. Alternatively we could start from a regular dodecahedron and glue a 5-tunnel Boy cap (Fig.12d) onto each one of its pentagonal faces (See 2013 Art Exhibit). Some of these models might make nice constructivist sculptures when realized as large tubular sculptures [14].

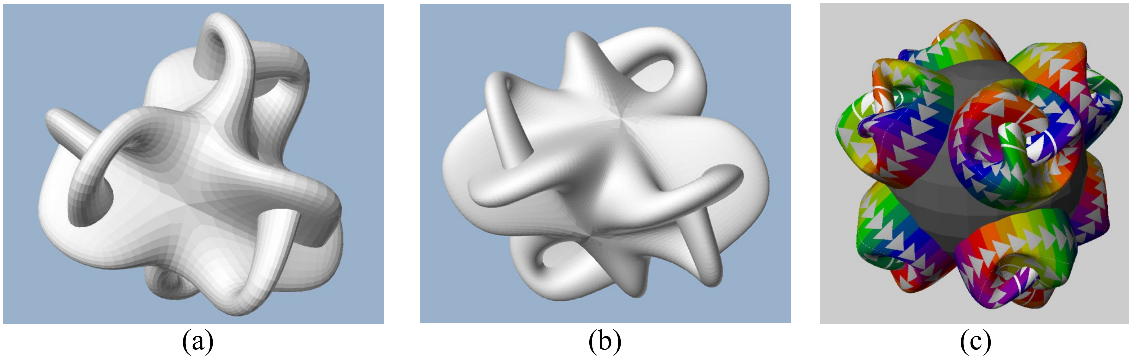


Figure 10: *Single-sided surfaces of higher genus: (a,b) genus-4, two different relative rotations; (c) genus-8 surface with the symmetry of an oriented octahedron.*

4. Building Arbitrary Surfaces

Now the moment has come to look at different classification schemes in more detail. Two compact surfaces are *topologically equivalent* if both are orientable or both are single-sided, and if they agree in their Euler characteristic and in the number of punctures or "rims." (E.g., Fig.11a exhibits four rims).

Following the didactic approach of my last two Bridges papers, I will try to illustrate this by taking a constructive approach. To obtain a topological model with a desired Euler characteristic, χ, we proceed as follows: We start with a sphere and assume that we have drawn a reasonably detailed mesh on it, so that we have many more facets than the genus of the surface we are interested in. Regardless of how we draw this mesh on the sphere, we will find that the Euler characteristic $\chi = V–E+F = 2$. For every isolated face that we punch out, χ will decrease by one; thus we just need to punch out the exact number of faces (Fig.11a) that yield the desired Euler characteristic!

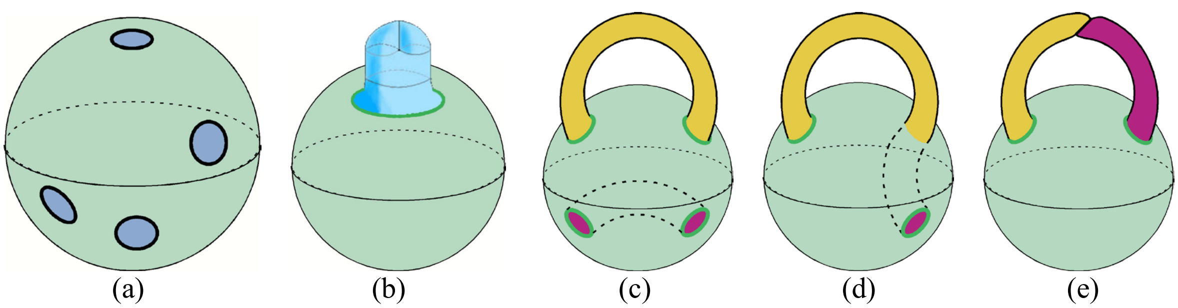


Figure 11: *Constructing a surface of Euler characteristic* χ*=2–h*, *by punching* ***h*** *holes into a sphere (a) and inserting either (b) cross-caps or Boy caps, or (c) handles, or (d,e) cross-handles.*

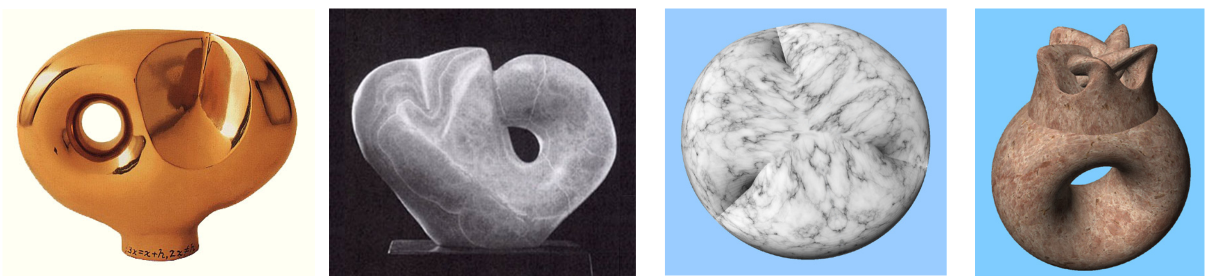
Now, to obtain a <u>closed</u> surface, we need to repair all the openings that we have punched by gluing some other surface elements into those holes. These are the repair patches that we may consider:

A **Boy cap** or **cross-cap** (Fig.11b): This provides a passage from one side of the surface to the other; just a single such element would render the surface single-sided and thus non-orientable. It turns out that both of these elements by themselves do have $\chi = 0$, and in the insertion process they coincide with the same number of edges and vertices along the rim of a hole. Thus their addition does not change the Euler characteristic of our surface under construction, no matter how many such caps we add to holes already punched. In our context we prefer to use the Boy cap (Fig.2c), since this is a smooth, immersed piece of surface, as required for the study of regular homotopies.

Handles (Fig.11c): We can deal with two holes simultaneously and add a *handle* between them. This is a piece of tubing, which we glue with one end into one hole, and with its second end into another hole (Fig.11c). The open-ended tube segment by itself also has $\chi = 0$, and gluing it to two holes will not change the Euler characteristic of our surface. But there is one possible change such a handle can make. If we glue its two ends onto the surface from different sides, letting the tube pass through the surface somewhere else without making a topological connection (Fig.11d), then we force the surface to become non-orientable; such a handle is called a **cross-handle**. Another way to make a **cross-handle** is to insert a pinch line in the tube (ending in two Whitney umbrellas) where the handle-surface passes through itself; such; this cross-handle would then be attached with both ends to the same side of the surface (Fig.11e).

Thus we have some flexibility how we want to do this "plumbing" to close up all the punctures that we created to obtain the right value for χ. But we can be restrictive and use mostly just plain tubing to form regular handles. A single-cross handle readily makes a Klein bottle; and after that all we need to do is add regular handles, with possibly one more Boy cap if the number of holes we need to fill is odd.

Helaman Ferguson has created several sculptures to depict such modular constructions of higher-genus surfaces (Fig.12a,b): *Torus with Cross-Cap* (1989) and *Eine Kleine Rock Musik III* (a Klein bottle plus a cross cap, 1986) both depict non-orientable surfaces of genus 3. That same topological surface, also known as Dyck's surface [5], can also be formed as a connected sums of 3 cross-caps (Fig.12c), or of a torus and a Boy cap (shown with 5 tunnels in Figure 12d). All these surfaces are topologically equivalent!



(a) (b) (c) (d)

Figure 12: *Single-sided genus-3 surfaces: (a) "Torus with Cross-Cap" (b) "Eine Kleine Rock Musik III" – both courtesy of Helaman Ferguson. (c) Three cross-caps; (d) 5-tunnel Boy cap on a torus.*

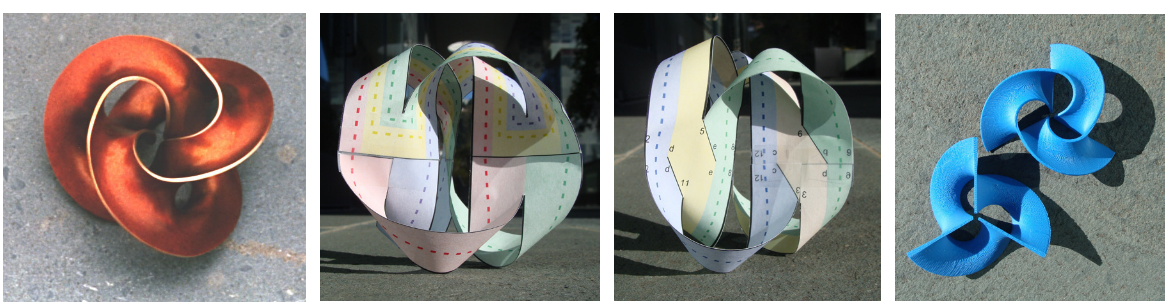
We can also apply a more discriminating classification by asking whether two surfaces can be smoothly deformed into one another without creating any tears or singular points with infinitely high curvature. To start with the two surfaces have to be smooth and topologically equivalent. But there are two types of Möbius bands, left-twisted and right-twisted ones, and correspondingly there are left- and right-twisting Boy caps. While these are topologically equivalent, they cannot be smoothly transformed into one another, and they are thus in different regular homotopy classes [16]. There are also two structurally different tori, the ordinary donut (**TOO**), and the fully twisted loop with a figure-8 profile (**T88**) [15]. Therefore, when we construct surfaces of higher genus, we have to be conscious of the exact type of building blocks we are using to figure out what regular homotopy class the result will belong to. Most relevant to our current context, Pinkall [9] shows: All orientable surfaces of a given genus > 0 can be composed of tori, at most one of which needs to be twisted. Similarly, all single-sided surfaces are regularly homotopic to a connected sum of Boy surfaces (projective planes). If we try to minimize triple points, then we can decompose such surfaces also into a connected sum of ordinary tori plus one of the following eight surface elements: **K8L**, **K8R**, **KOJ**, **KOJ#T88**, **BL**, **BR**, **K8L#BL**, or **K8R#BR**. – Thus all surfaces can be modeled in a regular homotopic way with at most **one** triple point (from one of the last four elements). This assumes unmarked, untextured surfaces with no explicit coordinate system given.

5. Sculpture Analysis of Surfaces with Boundaries

All of these surfaces can, of course, be punctured and then have one or more boundary components or rims. This allows such surfaces of higher genus to be embedded in Euclidean R3 space without any self-intersections. It also leads to a much larger variety of visually interesting sculptures, since the openings allow us to see part of the inside of these sculptures. Many artists have explored this domain. Among my heroes are Max Bill, Brent Collins, Eva Hild, … Given such a sculpture, it is often a non-trivial task to determine its genus or its Euler characteristic and to figure out how it can be decomposed into a connected sum of pure annuli and/or Möbius bands. I will give a glimpse of this problem with two examples: *Tripartite Unity* by Max Bill is clearly *single-sided*; the surrounding of the round opening at the bottom by itself forms a Möbius band (Fig.13a,b). Next we follow every rim (the hole boundaries) and determine how many separate closed contours there are; in this sculpture there is only a single rim! Next we need to determine the *Euler characteristic* χ. A convenient way to do this is to cut enough of the ribbons so that there are no more loops, and the sculpture decomposes into one or more topological disks (which here, however, look more like stars or spiders). To determine χ, we count number of disks and then subtract the number of ribbons that we have cut; the result here is: $\chi = -2$. From this we can then determine the genus of the surface with the formula: $genus = 2 - \chi - rims$. Thus *Tripartite Unity* is a single-sided surface of genus 3 bounded by a single rim and therefore equivalent to the connected sum of three Boy surfaces (also known as Dyck's surface [5] Fig.12), but with a single puncture added. Thus it should be possible to decompose this sculpture into 3 Möbius bands. I have succeeded in doing this with a paper model of this sculpture (Fig.13c). It turns out that a topologically equivalent structure (Fig.13d) is also depicted in the *Topological Picturebook* [4] (but the linking of the three bands differs). A somewhat different analysis – but with the same result – is given by [8].

(a) (b) (c) (d)

Figure 13: *Analysis of "Tripartite Unity" by Max Bill: (a,b) metal sculpture, (c) a paper model of this sculpture showing 3 Möbius bands, (d) a topologically equivalent surface [4].*



(a) (b) (c) (d)

Figure 14: *Scherk-Collins toroids: (a) Minimal Trefoil, (b) decomposed into 4 Möbius strips, (c) decomposed into 2 Möbius strips and a 2-sided annulus; (d) 2 triply twisted Möbius strips.*

Other sculptures worth analyzing are the *Scherk-Collins Toroids*. My *Minimal Trefoil* (Fig.14a) comprises a sequence of three biped saddles connected into a twisted toroid. It also has a single boundary, but $\chi = -3$; and thus its genus is 4. The paper model depicted in Figure 14b shows that this surface can indeed be split into 4 Möbius bands. However, in this decomposition the 4 bands are not all equivalent: one of them (the yellow one) does not touch the rim of the sculpture at all. At this point I don't know whether it is possible to find a different solution in which all four Möbius bands are topologically equivalent. Since a topologically equivalent surface can also be constructed from two Boy caps on a torus with a single puncture, a decomposition into an annulus plus two Möbius bands is also possible; this is shown in Figure 14c. Cutting the Trefoil in its equatorial plane produces two triply twisted Möbius bands with three connections between them (Fig.14d). Other sculptures by Collins and Séquin present even more formidable challenges: e.g., the *Heptoroid* [12] is equivalent to a disk with 22 cross-caps grafted onto it, and thus it should be decomposable into 22 Möbius bands, or into 8 Möbius bands plus 7 annuli.

6. Whimsical Applications and Work in Progress

Here we discuss some possible projects in which these low-genus, single-sided surfaces are used as design elements to create some unusual or even funky artifacts.

Furniture: In Bridges 2000 I used the geometry of the Möbius band to create unusual designs for bridges and for buildings [13]. Here I take the opportunity to extend that list with some furniture pieces. *Möbius Chair* (Fig.15a,b) is a variant of the simple chair formed from a single ribbon by adding a twist. *Möbius Bench* is a twisted prismatic ribbon with a profile in the shape of a cross [2]. This inspired me to sketch out the *Klein Kouch,* which is based on a 3-mouth Klein bottle with a figure-8 profile (Fig.15d).

(a) (b) (c) (d)

Figure 15: *Furniture based on non-orientable surfaces: (a, b) Möbius Chair, (c) Möbius Bench by [2], (d) Klein Kouch – topologically just a Klein bottle.*

Hats and Caps: There are quite a few knitted Klein-bottle hats to be found, e.g. [1] (Fig.16a). In the same vein, inspired by the pope's "divinity hat", the geometry of the cross-cap could make an "infinity" cap; a prototype was knitted by Margareta Séquin (Fig.16b). Also, as the name implies, a Boy cap could also serve as the model for a knitted cap or a woven hat (Fig.16c). I am not a knitting expert, so others will have to figure out how to knit these shapes in the most efficient ways. I can see ways in which this surface geometry could be stitched together from several individual patches – similar to the way that I have constructed this shape from several Bézier or B-spline patches (Fig.7c).

(a) (b) (c)

Figure 16: *(a) Klein-bottle hat [1]; (b) woolen cross-cap skullcap; (c) a virtual Boy cap straw hat.*

Cups and Mugs and More: Turning a cap upside down transforms it into a cup. Thus a solid, ceramic version of the shapes discussed in the previous section should readily make intriguing coffee cups (Fig.

17a) or beer mugs (Fig.17b) for geometry afficionados. The complete Boy surface realized as a hollow water-tight shell can yield a floating device that could serve as a buoy. The anchor chain could be attached to the single "pole" of this surface. Each of the three tunnel lobes could be outfitted with a solid ring to which one or more boats could then be tied (Fig.17c).

(a) (b) (c)

Figure 17: *(a) 3-tunnel Boy cup and (b) 5-tunnel Boy stein; (c) Boy buoy.*

7. Conclusions and Future Work

The last two sections are just glimpses into fertile domains for study that are practically limitless. In my talk I can show more objects than I can depict in this paper, and hopefully this will stimulate others to come up with further outlandish designs and artistically stimulating objects. A more systematic treatment of topological surface equivalence and an analysis of a broader range of abstract geometrical sculptures will be the subject of a forthcoming paper.

Acknowledgements

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