The Beauty of Equations

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Abstract

This paper discusses grounds for ascribing beauty to equations. Philosophical accounts of beauty in the context of science tend to stress one or more of three aspects of beautiful objects: depth, economy, and definitiveness. The paper considers several cases that engage these aspects in different ways, depending on whether the equation is considered as a signpost, as a stand-in for a phenomenon, as making a disciplinary transformation, or as inspiring a personal revelation.

Introduction

Certain equations have become more than scientific tools or educational instruments, and exert what might be called cultural force. A few equations have acquired celebrity status in that people recognize them while knowing next to nothing about them, while other equations attract fascination and awe because they are thought to harbor deep significance. For these reasons, equations are frequently encountered outside science and mathematics, and even turn up in novels, plays, and films, having effectively acquired the status of cultural touchstones. Sometimes what the public encounters in the form of an equation is simply an ordinary thought cloaked in mathematical dress. Even the form of certain equations can be culturally influential, as when instructions for behavior are presented as equations.

Several recent books have broached the subject of the meaning, greatness, or aesthetic value of certain important equations [1-5]. I want to discuss the last of these subjects: aesthetics. Scientists do sometimes ascribe beauty to theories or equations, and occasionally declare beauty to be an essential property of fundamental equations. Paul Dirac's remark that "the *only* physical theories that we are willing to accept are the beautiful ones" is the most well-known, if enigmatic, of such declarations [6]. But aren't equations functional, a means of computing or acquiring data rather than something to linger over aesthetically? When scientists call equations beautiful, aren't they using the word loosely or equivocally? Many of the just-referenced books answer with metaphors. Farmelo, for instance, compares equations to poems, on the grounds that neither great equations nor great poems can be altered without spoiling them, and also because an equation, like a poem, is a "concise and highly charged form of language"—although Farmelo then goes on to describe differences between equations and poems [2]. Guillen also compares equations to poems but on different grounds, saying that equations "state truths with a unique precision, convey volumes of information in rather brief terms, and often are difficult for the uninitiated to understand." He adds, "And just as conventional poetry helps us to see deep within ourselves, mathematical poetry helps us to see far beyond ourselves" [3].

Is it possible to call equations beautiful in a non-metaphorical sense; that is, to mean that equations are *really* beautiful rather than just *like* things that really *are* beautiful? Philosophical accounts of beauty in the context of science stress three possible aspects of beautiful objects: depth, economy, and definitiveness [7]. In one line of thinking that stretches from Plato to Heidegger, the beautiful is intrinsically connected with the fundamental, and the beautiful thing is that which points beyond itself to the true and the good. Another line of thinking, exemplified by Aristotle, focuses more on the composition of the beautiful object, emphasizing symmetry or harmony and the fact that nothing could be added or taken away from the object without interfering with its beauty. Yet another group of thinkers, who include David Hume and Immanuel Kant, stress the role of satisfaction: the beautiful object incites in us a feeling of pleasure, and the realization that that we need seek no further, for this is what we wanted all along even if we did not know beforehand. This paper discusses several ways that equations might be described as beautiful, though in the cases I discuss the three aspects I just mentioned – depth, economy, and definitiveness – apply in different ways and sometimes not well.

Graphic Signposts

The images in Figure 1 are from a hand-bound, limited edition book *Equations*, by the British artist-designer Jacqueline Thomas, of the Stanley Picker Gallery at Kingston University [8]. Thomas was inspired to create these images, she says, because she felt that certain fundamental equations "are visually and graphically 'beautiful' to look at," and decided to accent their beauty by setting them alongside another form of graphic beauty that she discovered in a 19th-century book, written and illustrated by a school teacher, about sciography—the projection of shadows—for a course in the subject intended for students in engineering or architectural draughtsmanship [9]. Thomas tried to balance graphics and images, matching more or less complex equations with more or less complex graphics, fitting them together in a puzzle-like fashion until the result looked right to her. She says, "The visually complex graphics of the Schrödinger equations [see Figure 1, bottom], for example, are reflected in the energetic geometry of the image," so that the page containing them has a dynamic quality, "where drawings, numbers, symbols, and brackets together make up a balanced page."

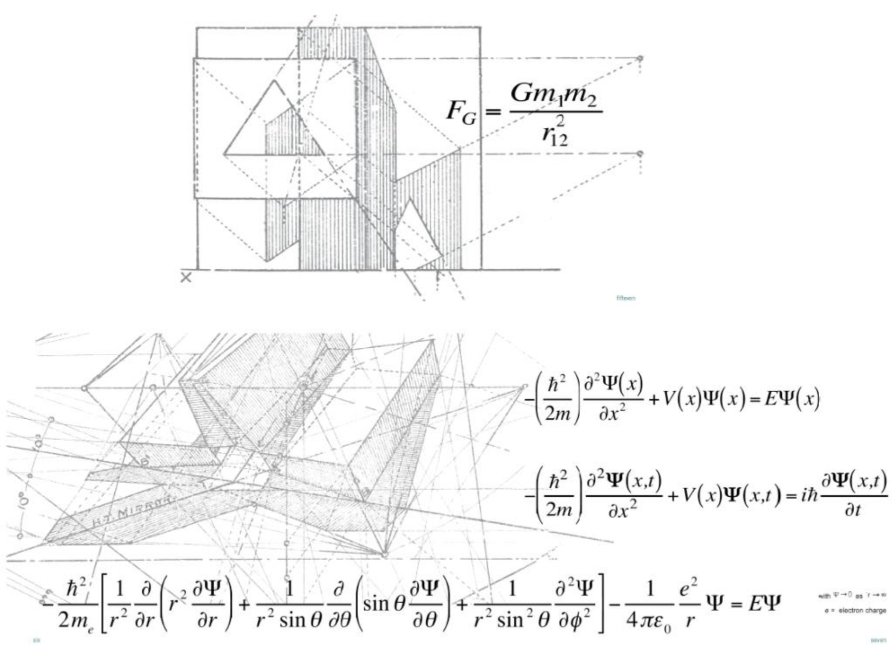

Figure 1: *Two images from Equations, a hand-bound book, courtesy Jacqueline Thomas*

Thomas's book embodies an unusual approach to the beauty of equations, in that it is tied to a specific graphical representation. An equation, though, is a particular way of describing a physical phenomenon from within a mathematical language, an inherited, contingent, and frequently changing form of representation. Many familiar equations sometimes deemed to be beautiful, including the famous $E = mc^2$ and Maxwell's equations, were originally written down with different symbols and in a different form, while others, including Newton's $F = ma$ and the equation expressing his law of universal gravitation (see the top of Figure 1), were not even originally written as equations at all.

Suppose one were to present the artist with two very different versions of the 'same' equation and then (a) see whether and how much the resulting geometric forms would differ, and (b) examine the artist's "outtakes" (preliminary sketches) to see what different geometric forms had been inspired by the same graphical representation of an equation. What this might reveal is to what extent the artist is responding to the typographic qualities of the equations, or to the intrinsic mathematical or physical principles. When I put this question to Thomas, she said that different versions of an equation would prompt her to use different forms. "With geometric images," she said, "the 'architecture' of the numbers and symbols seems to dictate how they should sit in context on the page."

Other examples of art based on the visual appeal of purely graphic representation of equations include Bernar Venet's "equation paintings" (at http://www.bernarvenet.com/), and a London billboard display in 2012 of Schrödinger's equation (http://www.scienceomega.com/article/671/schrodingers-billboard-aims-to-make-londoners-curious). But it is hard to see how to apply the three aspects of beauty to these instances; the beauty of the graphic representation does not seem to be part of the aesthetic. That would be like calling a signpost beautiful simply because it points to a beautiful object.

Phenomena

A different route to the beauty of equations is to attribute it to the phenomenon described rather than to the graphical representation of the equation itself. Here Maxwell's equations provide a good example. In one tribute to them, Feynman declared that "Maxwell's discovery of the laws of electrodynamics" from the long view of human history, will be judged to be "the most significant event of the 19^{th} century," with the American Civil War fading into "provincial insignificance" by comparison [10].

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0$$

$$\mu\alpha = \frac{dH}{dy} - \frac{dG}{dz}$$
$$\mu\beta = \frac{dF}{dz} - \frac{dH}{dx}$$
$$\mu\gamma = \frac{dG}{dx} - \frac{dF}{dy}$$

$$P = \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dz}$$
$$Q = \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy}$$
$$R = \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz}$$

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p' \qquad p' = p + \frac{df}{dt}$$
$$\frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 4\pi q' \qquad q' = q + \frac{dg}{dt}$$
$$\frac{d\beta}{dx} - \frac{d\alpha}{dy} = 4\pi r' \qquad r' = r + \frac{dh}{dt}$$

$$P = -\xi p \quad Q = -\xi q \quad R = -\xi r$$

$$P = kf \quad Q = kg \quad R = kh$$

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Figure 2: *Two versions of Maxwell's Equations*

Feynman, this time, was not joking. But neither he, nor others who have characterized Maxwell's equations as beautiful, had in mind the specific equations that Maxwell himself set down. These appear in a chapter of Maxwell's *Treatise* (1873) entitled "General Equations of the Electromagnetic Field." In it, Maxwell summarizes his work in twelve steps, each involving an equation or group of equations – 20 in all – that lay out "the principal relations among the quantities we have been considering." These equations not only look much different from the ones now familiar, but are based on a different set of concepts, such as **A**, the vector potential, and ψ, the scalar potential. A decade later, Oliver Heaviside reformulated Maxwell's work. Using **D** and **B** to represent current, and **E** and **H** to represent the electric and magnetic forces, he reduced the number of related equations to just four, so concise that they can be found printed on T-shirts and coffee cups. Only by drastically changing their form of presentation, Heaviside once remarked, was he first able to see the phenomenon clearly.

From this perspective, what is beautiful is not the graphical form of the equations – where it matters whether there are four or twenty equations, for which the sciography would be different – but rather the phenomenon that Maxwell and then Heaviside (more concisely) described with them: the electromagnetic field. What's beautiful about this? First of all, it was new, completely unanticipated by Newtonian mechanics, and Maxwell's equations characterize this new phenomenon fully. Second, like Dirac's equations, Maxwell's equations contain a prediction for something that went far beyond the existing theory that they were apparently only summarizing; namely, the existence of electromagnetic waves. Finally, the understanding of electromagnetism that grew out of these equations helped transform electromagnetism from a curiosity into the technological foundation of the electronic age; they are fundamental both theoretically and technologically.

This is a quite different approach to the beauty of equations than trying to find that beauty in an equation's graphic form. The difference between attributing beauty to a graphical representation of a phenomenon and the phenomenon with which it is associated is somewhat like the difference between paying attention to a signpost that guides us to the Grand Canyon or to the Mona Lisa, and to these objects themselves. Here the beauty of Maxwell's equations is more than that of a nice signpost. Their beauty seems to derive from the way these equations show us something fundamental about the world itself. Moreover, Heaviside's reformulation is a lot more economical than Maxwell's original version, and was much easier for electricians (who formed much of his intended audience) to use. Heaviside's version of Maxwell's equations seems to engage all three of the above-mentioned aspects of beauty.

Disciplinary Transformations

Even phenomena change over time, however. In the course of science history one phenomenon proves to be not as independent from others as originally thought, but part of a bigger whole or wholes. Magnetism proves to be an aspect of electromagnetism, which in turn proves to be an aspect of the electroweak force, and so forth. The beauty of some equations has been associated, not so much either with their graphical representation or with the phenomena with which they are associated, but with their role in transforming science and math. In such transformations they not only simplify and clarify, but may also close off one chapter in science history and open others.

$$e^{i\pi} + 1 = 0$$

Figure 3: *Euler's Identity/Equation*

Euler's equation $e^{i\pi} + 1 = 0$ is technically an identity rather than equation because it contains no variables–-though it contains five of the most fundamental concepts of mathematics and four operators, each exactly once. Still, it is often referred to as an equation, and as an especially beautiful one. Keith Devlin has written that "this equation is the mathematical analogue of Leonardo da Vinci's Mona Lisa painting or Michelangelo's statue of David," while Paul Nahin has written that the equation sets "the gold standard for mathematical beauty" [11]. Metaphors, again. I do not object to these metaphors as such, but what I think is interesting is why they work as metaphors—in what gives beauty to equations, rather than the beautiful things that equations resemble.

To outline the way I understand why people call Euler's identity beautiful let me abbreviate my discussion with a metaphor: Mathematics often grows in an indirect way, in the way that many cities do [1]. Certain scattered settlements spring up first, with little interaction among one another. These settlements eventually cluster around one another, becoming neighborhoods, but because they form almost at random they are poorly adapted and little commerce takes place. A visionary leader emerges who understands each neighborhood, and by renaming some streets and building others between key centers and adding new buildings allows the settlements to grow into a greater structure that is simultaneously more simplified, organized, and unified.

That is the role that Euler played in 18$^{\text{th}}$ century mathematics. At that time, mathematics had two well-developed neighborhoods, geometry and algebra. By the beginning of that century, mathematics was in the process of evolving a new neighborhood called analysis. Euler's book *Introduction to Infinite Analysis* (1748) not only developed and reorganized analysis, but by displaying certain of its connections with the other neighborhoods moved it into the center city of mathematics. One of the central landmarks was now Euler's discovery of a deep connection between exponential functions, trigonometric functions, and imaginary numbers. This connection—which effected a deep disciplinary transformation—is what is so vividly on display in the equation $e^{i\pi} + 1 = 0$.

In a sense, this equation is only one among thousands of steps in Euler's work. Yet some steps acquire and deserve special status. Certain expressions serve as landmarks in the vital and bustling metropolis of science, a city that is continually undergoing construction and renovation. These are expressions that preserve the work of the past, orient the present, and point to the future. Theories, equipment, and people may change, but formulas and equations generally remain pretty much the same. They are guides for getting things done, tools for letting us design new instruments, and repositories for specialists to report and describe new discoveries. They summarize and store, anticipate and open up.

Euler's equation is a particularly dramatic example of this process, an emblem of the way he had recast mathematics and rearranged its ontology, or of the way that it had assigned phenomena to different and distinct domains. Euler rearranged this ontology so that analysis was at the center, with geometry and algebra as neighborhoods. Looking backwards, mathematicians may take the latest organization as self-evident. Indeed, the mathematician Carl Friedrich Gauss is said to have remarked that anyone to whom $e^{i\pi} + 1 = 0$ is not obvious is not a mathematician. When you are fully literate, nothing comes as a revelation. Euler's work brought about a transformation and reorganization of mathematical knowledge, of which this equation is the most succinct expression.

As Devlin wrote, "Euler's equation reaches down into the very depths of existence. It brings together mental abstractions having their origins in very different aspects of our lives, reminding us once again that things that connect and bind together are ultimately more important, more valuable, and more beautiful than things that separate." The reason to attribute beauty to this equation, then, may have nothing to do with its graphic representation, nor even with the phenomenon which it describes, but to the way it succinctly displays a disciplinary transformation in mathematics.

A Platonist might object that Euler's identity, like all other truths of mathematics, was a discovery, something already existing *out there* that some mathematician inevitably would have come across. But one can leave aside discussion of the merits of Platonism and simply say that this identity is beautiful because of the way it marked a transition in the *knowledge* that *mathematicians* had of what was out there and how the various elements of mathematics fit together. Euler's identity is like, to use a phrase from Plato himself in the *Symposium*, one of the "rising stairs" that takes us "upwards." Its depth, economy, and definitiveness apply here, if not to the truth itself, at least to a step in our learning about the truth.

Euler's identity is certainly deep and definitive, and surely economical. It is, as Coxeter quotes Kasner and Neuman, "perhaps the most compact of all formulas … it appeals equally to the mystic, the scientist, the philosopher, the mathematician [12]. Yet a skeptic might protest that, just as for Maxwell's original equations, more economical ways exist to state the same truth. What about $e^{i\pi} = -1$, for instance? As a reviewer to this paper pointed out, this is how anyone who proves this identity hits on first, then adds 1 to both sides. Indeed, the reviewer continued, "the most concise, most mysterious, and (dare I say it) most unimaginable form of Euler's identity arguably is $i^i = .2078795….$;" that is, a real number. In this form is absent the other concepts and operators – but has the additional conciseness increased or detracted from the aesthetic value?

The answer is that the various aspects of truth that I mentioned at the beginning – depth, economy, and definitiveness – each have costs and benefits, and balancing them in a single work does not allow a single solution. Furthermore, the beauty connected with this identity may be less connected with a single truth, let's say $i^i = .2078795…$, than with the fact that it serves as a clear and concise example of what an equation and formula can do: show how what seemed to be disparate and even incompatible elements – rational, irrational, and imaginary numbers – are implicated in a unity, and it does so concisely, with few moving parts, so to speak. It simultaneously simplifies, organizes, and unifies. It brings what equations do out into the open. This, rather than $i^i = .2078795….$, is what's served by the conciseness of Euler's equation/identity $e^{i\pi} + 1 = 0$. We might say that it is an equation that shows what it is to be an Equation.

Personal Revelations

The disciplinary transformation brought about by Euler's equation affected the structure of mathematics from the perspective of its professional practitioners. Another kind of beauty refers to transformational moments from the point of view of the individual. Equations, that is, are first introduced by innovators such as Euler who write them down for the first time in one kind of intellectual adventure or journey. Then these equations are learned by others who follow. Those who follow are involved in another kind of journey, a product of schooling or accident or curiosity or intent. Equations may then become the focus of personal revelation.

The Pythagorean theorem is a classic example. Here we don't know who took the first journey. But we have countless stories of its rediscovery, both by people who learned it and by people who rediscovered it for themselves. These have sometimes been such powerful experiences as literally to have changed lives and careers. The power and magic of the Pythagorean theorem arise from the fact that, while complex enough that its solution is not apparent at the outset, the proof process can be condensed sufficiently to constitute a single experience. The British philosopher Thomas Hobbes, for instance, encountered it that way. In a friend's library one day he happened to glance at the proof in a copy of the Elements. At first Hobbes disbelieved it, but after following Euclid's presentation changed his mind. The experience of going from disbelief to belief so astounded Hobbes that it transformed his scholarship and writing. The lesson Hobbes took away was that he should begin with clear definitions of terms, then work out the implications in an orderly fashion. The proof taught Hobbes a new way to reason, and then how to present the fruits of his reasoning persuasively, so that these results seemed necessary and universal. In the process, Hobbes went from being a talented yet unoriginal scholar in the humanities to a leading political philosopher [1].

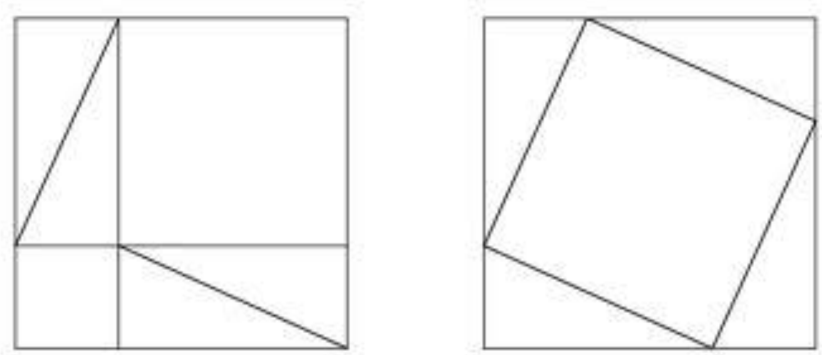

Figure 4: *Nonverbal proof of* $c^2 = a^2 + b^2$.

The term 'Pythagorean theorem' can refer to two things: a fact and a proof. The fact's discovery is ancient and its first discoverer unknown. The proof is the demonstration of how we know this fact, whose first manifestation (that we know about, at least) is in Euclid. When beauty is attributed to the Pythagorean theorem, it is generally connected with the impact that this proof had on the individual. From the individual's perspective, when it causes a personal revelation, the proof is deep, concise, and definitive.

Here, too, a skeptic might protest that this is a bad example because there are more powerful – deeper and more extensive – proofs, which might therefore be considered more beautiful if one considered aesthetic value to derive from depth alone. An example is the law of cosines, which covers all triangles, not just right triangles, and relates the lengths of the sides to the cosine of one of the angles; the Pythagorean theorem is just a special case of this law. Yet here, too, there is a tradeoff: the magic and beauty of the Pythagorean theorem is that its proof is concise enough to be mastered quickly by people like Hobbes who know no trigonometry. The theorem's proof – whether it be Euclid's or any of the other hundreds of ways that have been discovered – can provide an experience of personal revelation to each individual, thanks to the proof's depth, economy, and satisfaction.

Conclusion

Equations can act as signposts, as stand-ins for phenomena, as effecting disciplinary transformations, and as sparking personal revelations, and perhaps in other ways as well. Many of the traditional aspects of beauty can be applied to equations in each of these roles, with the possible exception of the first.

The issue is interesting because locating and exploring the affinities between the arts and the sciences is much more difficult than it appears. It is easy to be seduced by the inessential, to focus on surface similarities, and to come away from discussions entertained but not enlightened. The question of how beauty might be attributed to equations offers one way to begin to understand these affinities [13].

Acknowledgment

I would like to thank the reviewer of this paper whose comments were deep, economical, and definitive—and nearly as long as the original submission.

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