Bending Circle Limits

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Abstract

M.C.Escher's hyperbolic tessellations "Circle Limit I-IV" are based on a tiling of hyperbolic plane by identical triangles. These tilings are rigid because hyperbolic triangles are unambiguously defined by their vertex angles. However, if one reduce the symmetry of the tiling by joining several triangles into a single polygonal tile, such tiling can be deformed. Hyperbolic tilings allow a deformation which is called bending. One can extend tiling of the hyperbolic plane by identical polygons into tiling of the hyperbolic space by identical infinite prisms (chimneys). The original polygon being the chimney's cross section. The shape of these 3D prisms can be carefully changed by rotating some of its sides in space and preserving all dihedral angles.

The resulting tiling of 3D hyperbolic space creates 2D tiling at the infinity of hyperbolic space, which can be thought of as the sphere at infinity. This sphere can be projected back into the plane using stereographic projection. After small bending the original circle at infinity of the 2D tiling becomes fractal curve. Further bending results in thinning the fractal features which eventually form a fractal set of circular holes which in the end disappear.

1 Building the three dimensional chimney

Widely know tessellations of hyperbolic plane are generated by reflections in the sides of triangles with kaleidoscopic angles (in order to form kaleidoscope triangle's angles have to be π/n). In particular all four of M.C. Escher "Circle Limit I-IV" woodcuts are based on rotational subgroup of kaleidoscopic triangle groups.

Less known are tessellations by polygons with more then 3 sides. Such n-gonal tessellations are especially interesting because they can be continuously deformed in different ways, which we will illustrate and use here. Let us consider a 2223 kaleidoscopic quadrilateral in the *xy* plane of hyperbolic space. The quadrilateral has angles $(\pi/2, \pi/2, \pi/2, \pi/3)$. An example of such quadrilateral is shown in Figure 1 in the Klein-Beltrami model of hyperbolic space. In that model the whole hyperbolic space is represented as the interior of the unit ball. Planes are intersections of euclidean planes with the ball. Lines are intersections of Euclidean lines with the ball. The surface of the ball located at infinity of hyperbolic space. This model is not conformal (does not preserve the visual angles), but it is more suitable for drawing. Reflections in the sides of this 2223 quadrilateral generate a tiling in the *xy* plane.

Now we will extend the quadrilateral into infinite chimney as shown in Figure 2. The edges of the chimney are 4 infinite lines perpendicular to *xy* plane incident to the quadrilateral's vertices. The sides of the chimney are infinite bands which span pairs of sequential edges. If instead of reflections in the sides of the quadrilaters we use reflections in the sides of chimney, the tiling of plane by the 2223 quadrilateral will naturally extended to a tiling of the whole hyperbolic space by these infinite chimneys. Open ends of chimneys form tiling of the surface of the ball by curved quadrilaterals. In Figure 3 we only show tiling of the surface of the ball (sphere at infinity) by the lower opening of the chimney. The tiling fills exactly half of the sphere. The tiling of the sphere can be transformed into tiling of the plane via usual stereographic projection as shown in Figure 3.

It is remarkable that the stereographic projection of this sphere tiling is identical to the image of the two dimensional 2223 tiling of a hyperbolic plane realized in the conformal Poincare model.

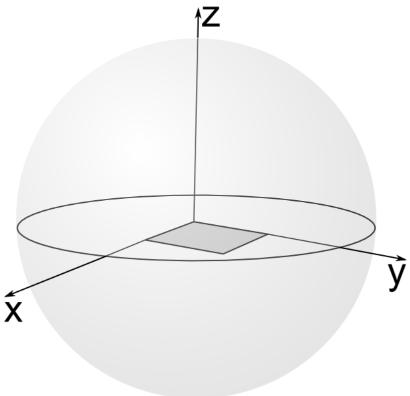

Figure 1: 2223 *quadrilateral in the XY-plane*

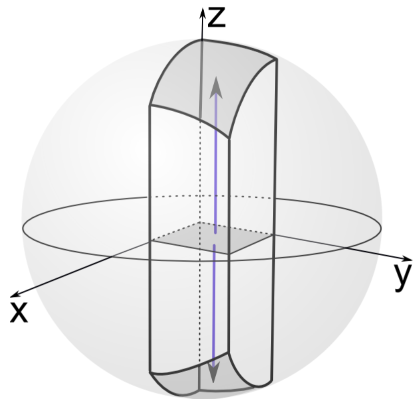

Figure 2: *Quadrilateral extruded into infinite vertical chimney*

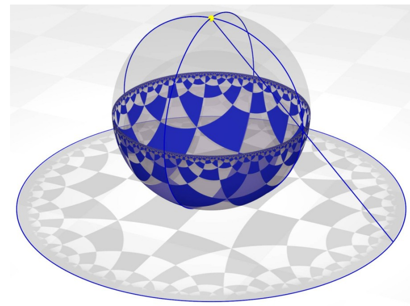

Figure 3: *Tiling of sphere and plane*

2 Bending the quadrilateral chimney

After we extended 2D tiling of the plane into 3D tiling of the space we have extra flexibility to explore how we can deform that 3D tiling. One of 3D deformations of the chimney is rotation of one side of the chimney around common perpendicular to its adjacent sides as shown in Figure 4. The side which is being rotated is highlighted. The axis of rotation in this case is x-axis. The top opening of the chimney becomes narrower and bottom becomes wider. We call such deformation bending.

It is important to note, that such bending is not possible in the euclidean space. We can form infinite euclidean tiling by infinite chimneys (parallelepipeds of infinite height), however the arbitrary small rotation of one side of the chimney will cause intersection of that side with the opposite side which in general breakes the tiling.

The tiling by slightly bent chimney from Fig. 4 is shown in Fig. 7. We show only the tiling formed by lower opening of the chimney. The tiles in the plane become slightly deformed. They are not anymore 2D hyperbolic polyhedron rendered in the Poincare model. Instead they are generic quadrilaterals whose edges are arcs of the circles. The circular boundary of the tiling from Fig. 7 (the limit set) has changed dramatically. It became fractal curve.

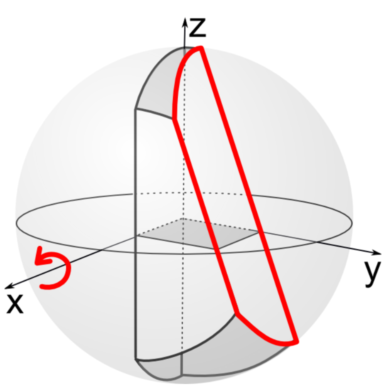

Figure 4: *Small bend*

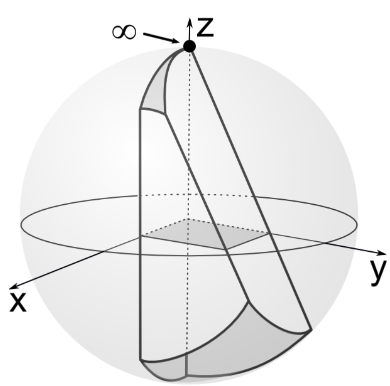

Figure 5: *Bend with cusp*

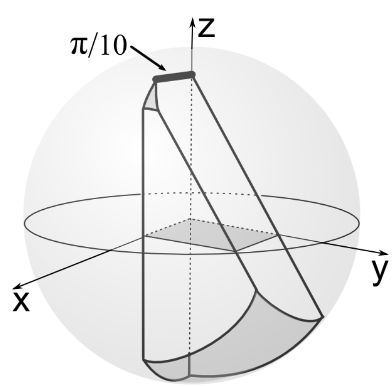

Figure 6: *Larger bend*

We can bend the chimney further to make rotated side of the chimney touch the opposite side in a point

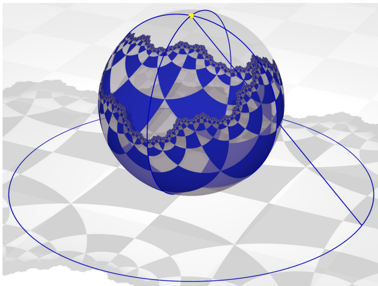

Figure 7 : *Tiling for Fig 4*

Figure 8 : *Tiling for Fig 5*

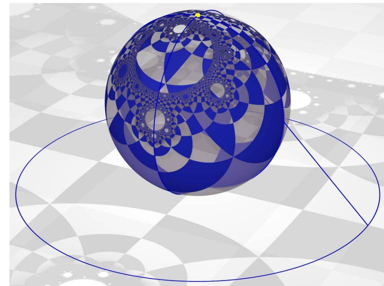

Figure 9 : *Tiling for Fig 6*

on the surface of the sphere (see Fig. 5). Hyperbolic planes touching at infinity are asymptotically parallel. The corresponding tiling on the surface of the sphere forms set of infinitely thin tentacles or cusps which appoach to that point of touch (see Fig. 8 and also Fig. 24 for a better view of the similar situation).

Further bend creates intersection of the opposite sides of the chimney (Fig. 6). The shape is not a chimney anymore. The bottom opening is still 2223 quadrilateral, but the upper opening becomes triangle and a new edge inside of hyperbolic space is created. The shape, which induces the tiling of the space via reflections has to have all dihedral angles to be kaleidoscopic (π/n). Therefore the dihedral angle at that new edge has to be kaleidoscopic as well. This condition limits the possible bend angles to a set of discrete points. In Fig. 6 that angle is $\pi/10$. The tiling of the plane corresponding to such tile is shown in Fig. 9. We still show only tiling by lower opening of the shape. The triangle at the upper opening of the shape has angles $\pi/2, \pi/3, \pi/10$ That triangle forms an infinite set of hyperbolic triangle tilings of different hyperbolic planes which plugs the circular holes of the tiling in Fig. 9. Fig. 15 and Fig. 16 show M.C. Escher tilings bent in the similar way.

We show only few examples of wide variety of possible deformations of the 3D hyperbolic tiling. Other way to look at the tiling generated by 4 reflections is to use 4 arbitrary planes in 3D hyperbolic space. These planes may intersect or be disjoint. The reflections in the planes form a tiling of the space, if each pair of planes has intersection angle π/n or is disjoint. The dimension of parameter space formed by such tiling can be calculated as follows. Each plane position has 3 degree of freedom, which gives 4×3 degree of freedom for non intersection planes. Motion of the plane set as a whole has 6 degrees of freedom. Therefore, we have left 6 degree of freedom for a set of non intersecting planes. Each intersection of planes subtracts one degree of freedom. There can be from 0 to 6 such intersections which we denote as N_i. Therefore the dimension of parameter space of tiling formed by reflections in 4 planes has dimension $6-N_i$, which varies between 0 for complete tetrahedron and 6 for a set of disjoin planes.

3 Making M.C. Escher patterns bendable

All four M.C. Escher "Circle Limit..." [1] woodcuts are based on subgroups of various hyperbolic triangle tilings. Triangles in the hyperbolic geometry can not be deformed without changing the angles. Therefore we need to reduce the symmetry of the tiling by joining few tiles together to obtain bendable tile.

Let's consider "Circle Limit I" pattern. The original pattern can be built from hyperbolic triangles with dihedral angles $\pi/3, \pi/4, \pi/4$ by reflections in sides a and b and half turn about center C of the third side (see Fig. 10). The triangle is rigid and can not be deformed. However we can join two such triangles together along side c and form quadrilateral $abde$ with dihedral angles $\pi/3, \pi/2, \pi/3, \pi/2$ (see Fig. 11). The complete tiling of the the hyperbolic plane is generated by 4 reflections in the sides a, b, d, e.

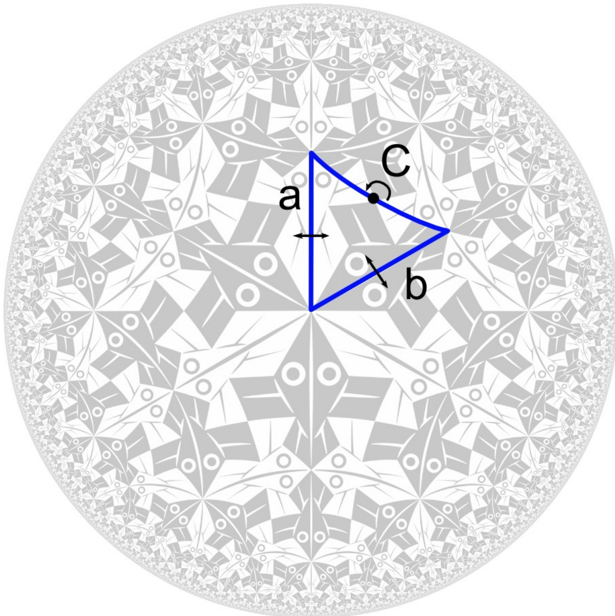

Figure 10 : *Original fundamental domain of the "Circle Limit I"*

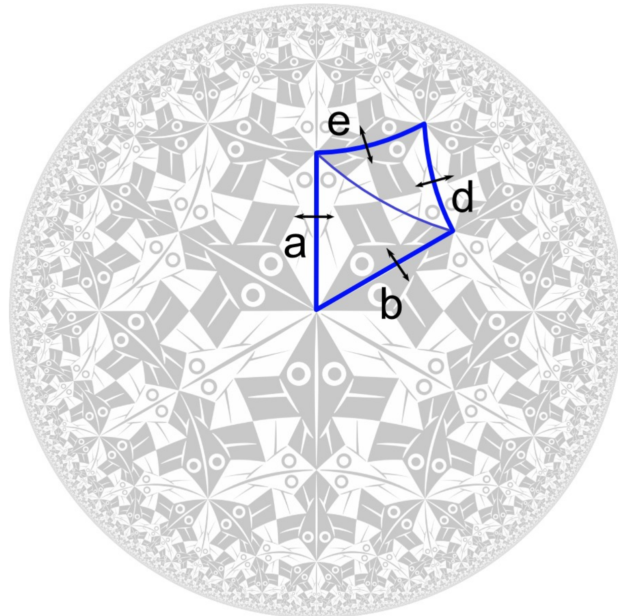

Figure 11 : *Bendable fundamental domain of the "Circle Limit I"*

We need to solve one more problem: how to transform pattern in the original fundamental domain into pattern in the deformed fundamental domain (see Fig. 12). The areas which we want to morph are bounded by segments of circle or straight lines. In principle it is possible to use brute force approach by covering the original domain with fine triangle mesh, stretching somehow the mesh onto deformed fundamental domain and interpolate images between small triangles of the meshes. This approach works in general situation, however it is computationally expensive and suffers from approximation of curved region with straight triangles. Instead we use the fact that our regions are formed by two pairs of hyperbolic planes forming opposite sides of the quadrilateral.

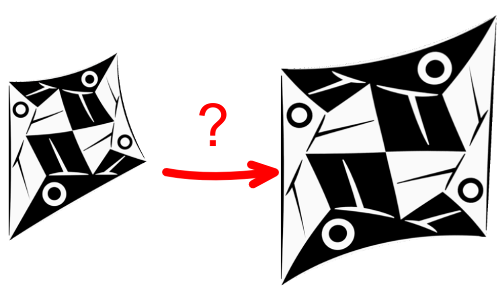

Figure 12 : *Pattern inside of the original fundamental domain has to be morphed into pattern in the deformed fundamental domain*

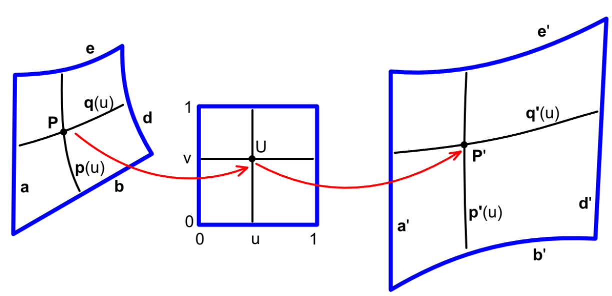

Figure 13 : *Morphing. Point P from the original quadrilatera is mapped onto point U in the unit square. Point U from the unit square is mapped into P' in the destination quadrilateral.*

We can create family of planes $\mathbf{p}(u)$ which interpolate between two opposite sides of quarilateral represented via 4D vectors $\mathbf{a}$ and $\mathbf{d}$ via following linear combination

$$\mathbf{p}(u) = (1-u)\mathbf{a} + u\mathbf{d}, u \in [0,1]$$

The combination is selected in such a way that $\mathbf{p}(0) = \mathbf{a}$ and $\mathbf{p}(1) = \mathbf{d}$. We have similar expression for family of planes $\mathbf{q}(v)$ which interpolates between another pair of opposite sides $\mathbf{b}$ and $\mathbf{e}$

$$\mathbf{q}(v) = (1-v)\mathbf{b} + v\mathbf{e}, v \in [0,1]$$

Each point P in the original quadrilateral $abcd$ belongs to one plane from family $\mathbf{p}(u)$ and one plane form family $\mathbf{q}(v)$ and is mapped to a unique point in the unit square via transformation which we denote T_1. Similar transformation T_2 maps each point P' in the deformed quadrilateral $a'b'd'e'$ into a point in the unit square. The concatenated transfromation $T_1 {T_2}^{-1}$ does the mapping form the original quadrilateral $abde$ into deformed quadrilateral $a'b'd'e'$. See Fig. 13 for illustration. Transfromation $T_2 {T_1}^{-1}$ does the reverse mapping of $a'b'd'e'$ into $abde$ which is needed for reverse pixel lookup described in the next section.

4 Rendering the tiling

There are two common approaches for rendering the tilings in the hyperbolic geometry. One that is used more often is to calculate some big the set of transformations of the tilings, transform fundamental domain using these transformations and render transformed tiles. The size of the image of the transformed fundamental domain may become very small and this can be used to truncate the invinitely large set of all transformations to some finite and manageable size. This approach works reasonable well in case of tiling of hyperbolic plane, where the tiles becomes smaller when tiles approach the circle at infinity.

In our case there is not circle at infinity and the boundary of the tiling can not be calculated explicitly. Moreover the size of transformed tiles can significantly increase. We use different approach by working in the image space and tracing each pixel back to it's pre-image in the fundamental domain. This approach called reverse pixel lookup was introduced in [2] for rendering two dimensional hyperbolic tilings. The reverse pixel lookup is directly generalizable to any number of dimensions of hypebolic space. We use such generalization to render images of tilings at the boundary of three dimensional hyperbolic space.

The idea of reverse pixel lookup is to apply a sequence of the generating transformations to each point in the image until the point is mapped into fundamental domain. The color of the mapped point in the fundamental domain is used to color the point in the rendered image. The transformation is added to the sequence if the current point and body of fundamental domain are on opposite sides of the corresponding face of the fundamental domain.

The usual computer graphics techniques such as antialiasing and mip mapping are used to increase the image quality and the rendering speed. The algorithm runs independently for each pixel and is also implemented to run on modern GPU. This allows real time rendering of the bent tilings.

5 Examples

Several examples are shown in Fig. 14–25 with explanations.

References

[1] M.C. Escher. *M.C. Escher, His Life and Complete Graphic Work*. 1992.

[2] M. von Gagern and J. Richter-Gebert. Hyperbolization of euclidean ornaments. *The Electronic Journal of Combinatorics*, 16R2, 2009.

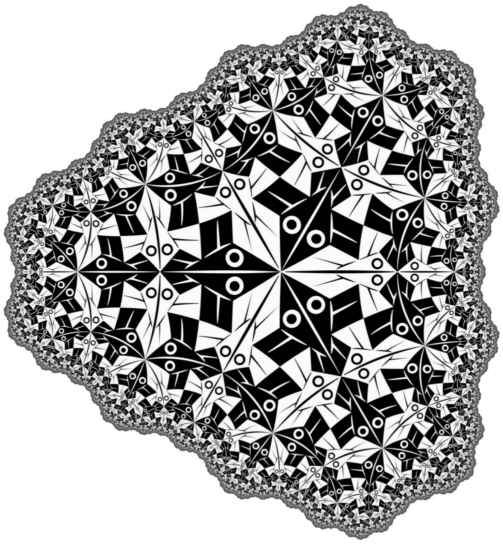

Figure 14 : *Small bend of "Circle Limit I" pattern. The original circular boundary becomes fractal curve. The dihedral symmetry *3 of the whole figure is preserved.*

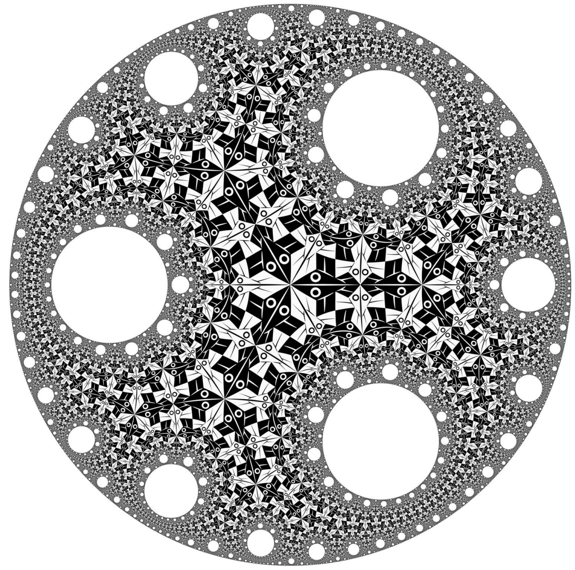

Figure 15 : *Larger bend of "Circle Limit I" pattern. The angle of intersection of opposite sides is* $\pi/16$*. The dihedral symmetry *3 is preserved.*

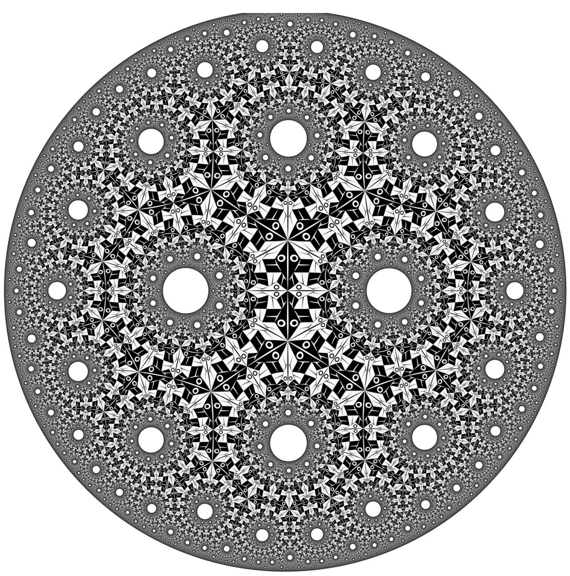

Figure 16 : *Larger bend of "Circle Limit I" pattern. The angle of intersection of opposite sides is* $\pi/8$*. The *2 kaleidoscope is moved to the center.*

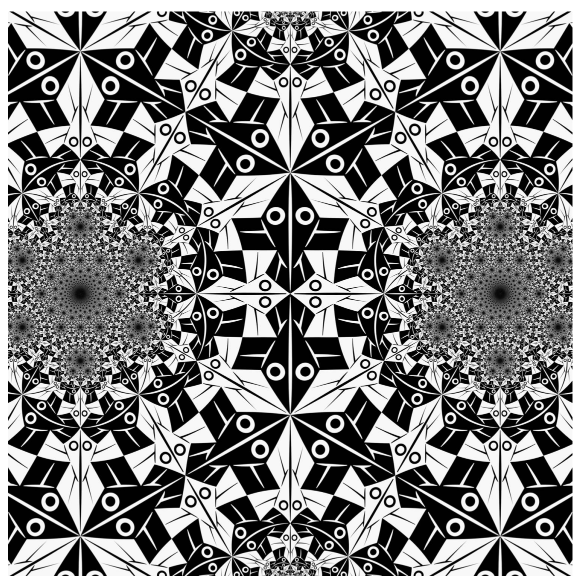

Figure 17 : *Critical bend of "Circle Limit I" pattern. Three sides of the chimney intersect in one point (cusp) at infinity and form *236 kaleidoscope. Compare with Fig. 16 to see the formation of the cusp.*

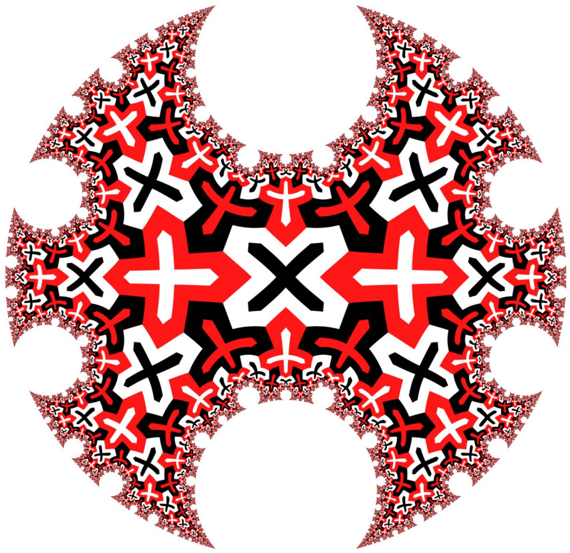

Figure 18 : *Bent "Circle Limit II" pattern. The center of the tiling is in the center of orignal pattern. The deformation reduces the symmetry in the center of the original tiling from* $*4$ *to* $*2$.

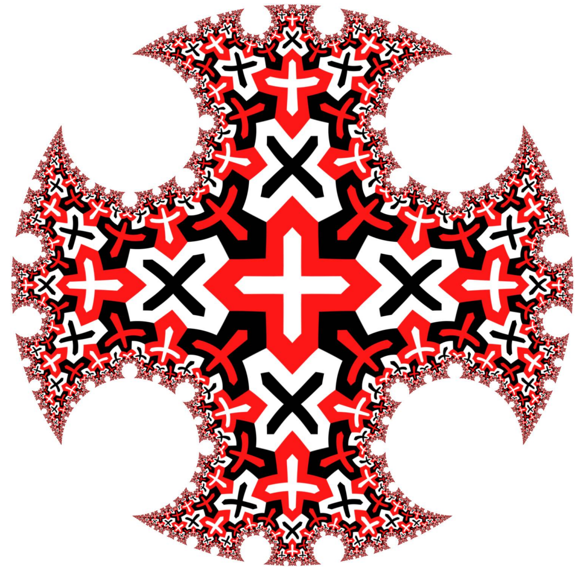

Figure 19 : *A variation of Fig. 18 with symmetry point* $*4$ *moved to the center.*

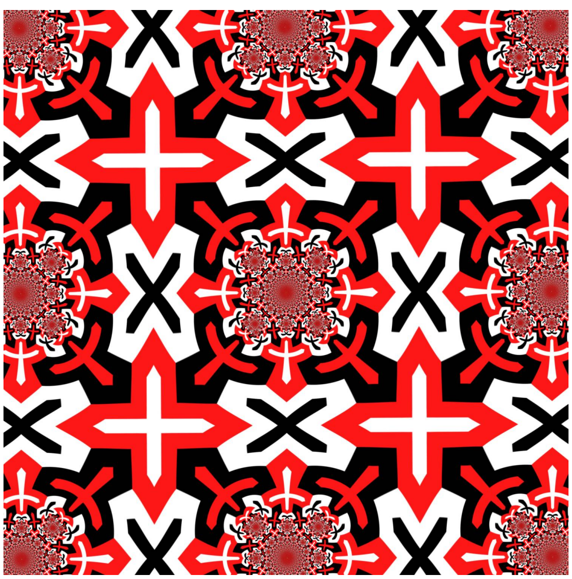

Figure 20 : *"Circle Limit II" pattern with cusp at the infinity of hyperbolic space. The symmetry of the tiling has subgroup with symmetry of eucludean plane kaleidoscope* $*442$.

Figure 21 : *Larger bend of "Circle Limit II" pattern. The upper opening of the chimney is completely closed. The fundamental domain has only one component on infinity - the bottom opening.*

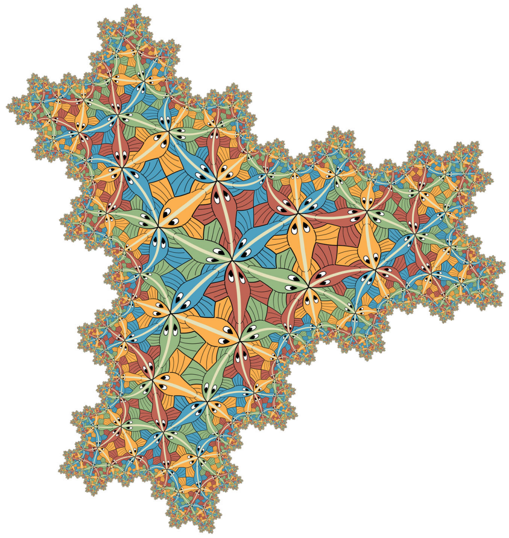

Figure 22: *"Circle Limit III" bend with fractal boundary with symmetry $*3$ moved to the center.*

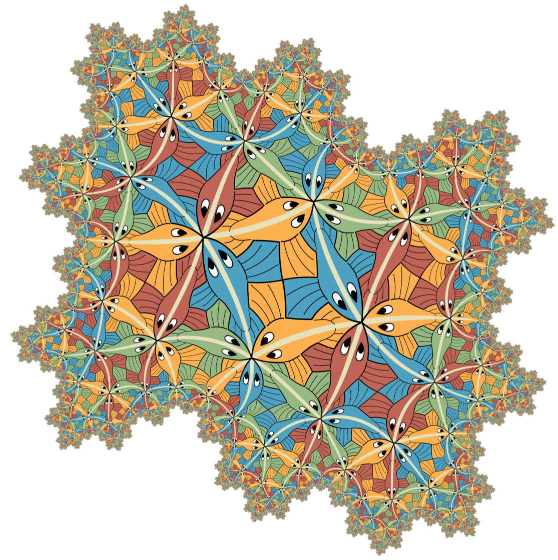

Figure 23: *The same bend as in Fig. 22 with symmetry $*2$ moved to the center. The exterior shape looks quide different, but is is the same same as in Fig. 22 only transformed by a motion of hyperbolic space.*

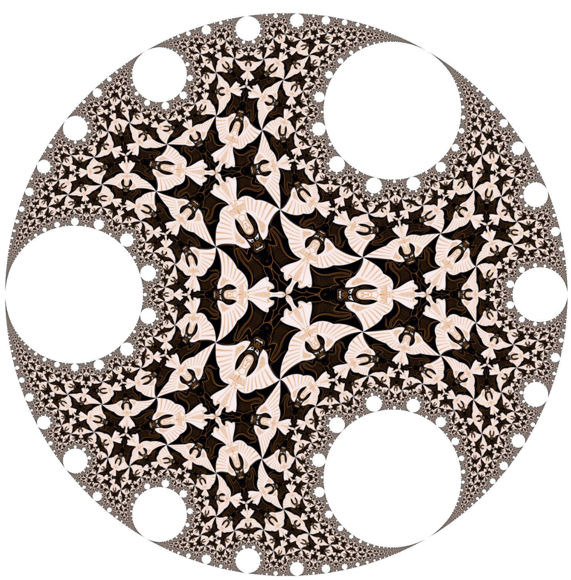

Figure 24: *"Circle Limit IV" bend with thin tentacles surrounding the cusp points.*

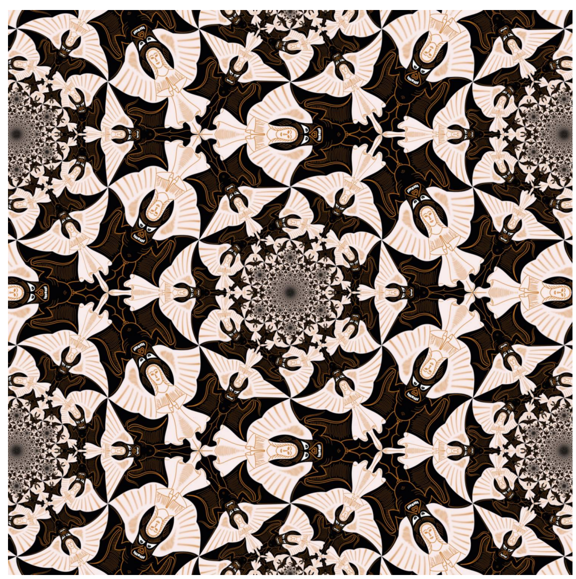

Figure 25: *"Circle Limit IV" bend with point of intersection of 3 sides located at the infinity of hyperbolic space. The symmetry of pattern has subgroup with euclidean symmetry $*333$*