Tiling and Tazhib of Some Special Star Polygons
A Mathematics and Art Case Study

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Abstract

This article presents the means for covering the interiors of some special star polygons using tiling and tazhib techniques. Several articles study approaches for tiling an unbounded plane using star polygonal motifs. However, the main focus of this article is to present methods for tiling bounded regions, such as the interiors of star polygons, using star polygons themselves to create self-similar designs. In addition, the paper introduces basic techniques of the classical Persian art form, tazhib. An artistic collaboration melding this art form with one of the tilings is included.

1. Introduction

It is believed that most medieval Persian mosaic designers were geometers as well. Because of this they were able to create sophisticated and elaborate tilings using star polygons. Among these star polygons were two, the pentagram and the decagram, that received more attention from the designers due to their mathematical properties, which are discussed in this article.

A pentagram can be constructed using the intersections of segments that constitute a (10, 3) star polygon. It also can be generated by the rotation of a 72° rhombus rotating around the vertex of the 72° angle.

A decagram can be created through the rotation of two concentric, congruent regular pentagons with a radial distance of 36° from each other's central angles.

To create interlocking star polygonal tessellations, a craftsman-mathematician would need to take long and careful steps to locate a fundamental region. Nevertheless, the interior regions of star polygons themselves can be considered as bases for interesting tiling and tazhib.

Tazhib (Illumination) is a classical Persian art form used for the decoration of books. In medieval Persia, the elegant and highly detailed art of tazhib was developed and its tradition has continued even today. In a traditional Persian tazhib, one can find mathematical ideas and concepts, such as symmetries, logarithmic and Archimedean spirals, polygons and star polygons.

In the next section a few star polygons will be introduced. In section 3 we study the ways that the interiors of some star polygons can be tiled. Section 4 is about the art of tazhib. In section 5 the step-by-step construction of a tazhib decagram will be shown. In section 6 an artistic collaboration of the author with a tazhib artist will be presented.

2. The Pentagram, Decagram, and Interlocking Polygonal Tiling

We use the two terms of pentagram and decagram for two non-convex polygons as follows: A pentagram can be made using the (10, 3) star polygon, a star that can be made by connecting every third vertex of a regular decagon. The intersections of the segments constituting the (10, 3) star polygon include the vertices of two concentric pentagrams with a radial distance of 36° (Figure 1). A pentagram can be divided into five rhombi; each consists of a pair of Penrose kite and dart (without special curves that make them quasi-periodic motifs). For a quasi-periodic tiling the special curves prevent a kite and a dart forming such a combination as a rhombus [9].

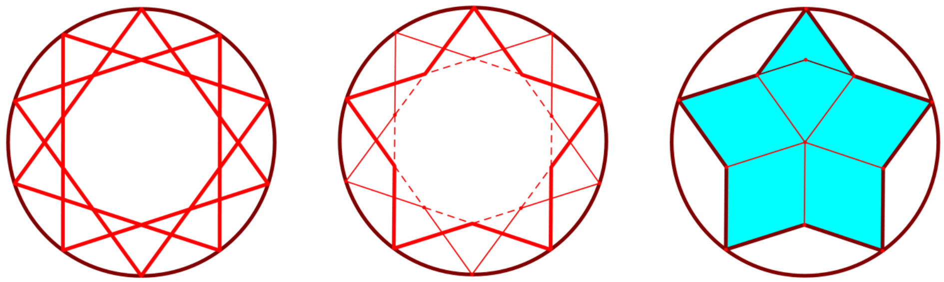


Figure 1: *The (10, 3) star polygon, two concentric pentagrams, and the pentgarm made from a rhombus.*

A decagram can be generated using two concentric congruent regular pentagons that have a radial distance of 36°, as is presented in the first two images in Figure 2. The last image in Figure 2 shows the relationship between the decagram and the (10, 3) star polygon. Moreover, it displays the connection between the10 kites surrounding the decagram and the (10, 3) star polygon.

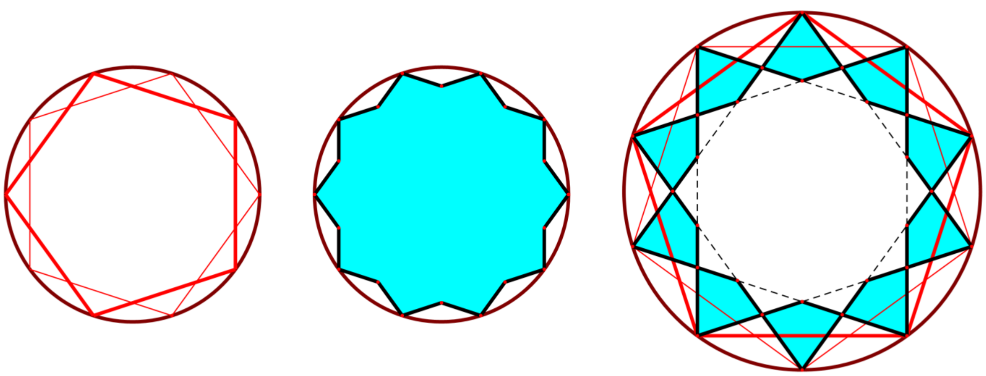


Figure 2: *Two concentric congruent regular pentagons, the decagram, and the (10, 3) star polygon.*

Figure 3 exhibits a tessellation with the pentagram motif (that also includes a decagram motif). Figure 4 presents four different tessellations that all have been constructed using the decagram as the main character among a total of five motifs.

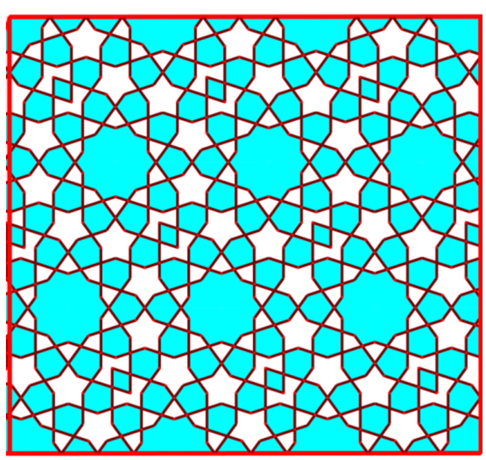


Figure 3: *An interlocking tessellation with the pentagram motif.*

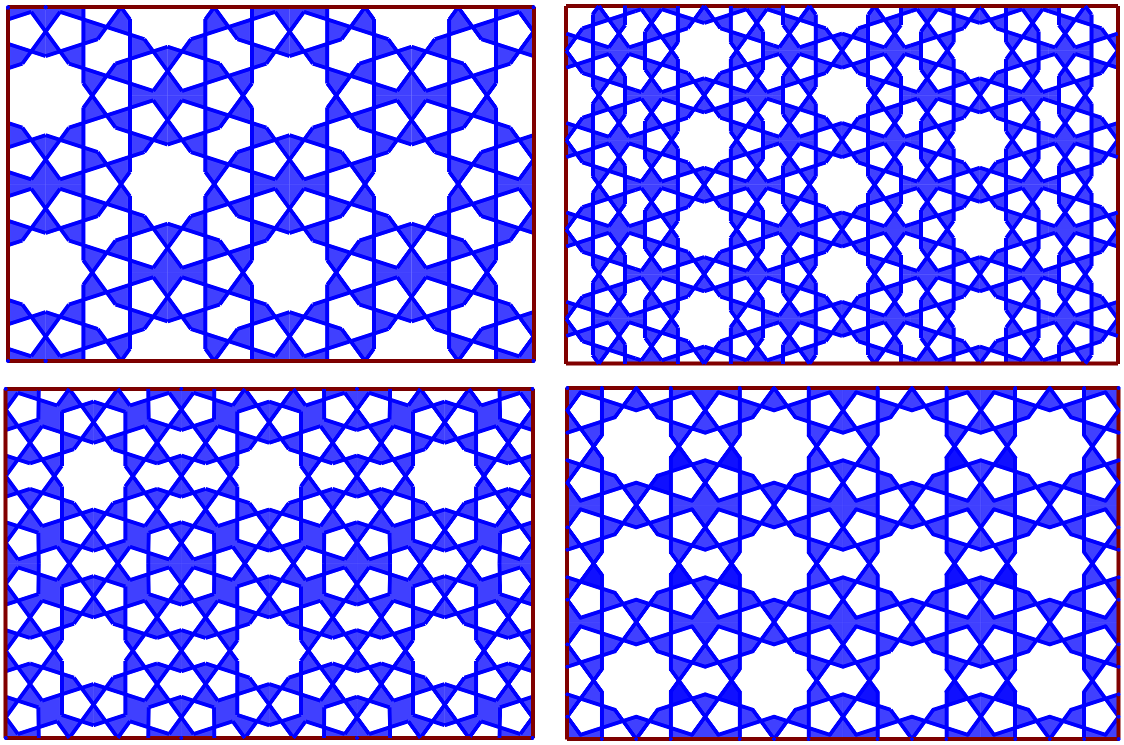


Figure 4: *Four different interlocking polygonal tessellations using the decagram as the main motif.*

Figure 5 shows all the motifs that have been used in the tessellations in Figure 4. In this article these motifs are called *sâzeh* (operative structure in Persian language) module tiles. These modules have their own specific Persian names: *Tabl* (decagram tile), *Sormeh Dân* (the bow tie tile), *Shesh Band* (the concave octagonal tile), *Torange* (the kite shape tile), and *Pange* (the pentagon tile) [9].

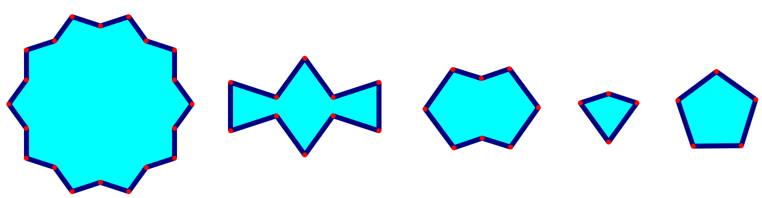


Figure 5: *The five sâzeh module tiles of Tabl, Sormeh Dân, Shesh Band, Torange, and Pange.*

Below we study patterns that cover the interiors of the pentagram and decagram in harmonious and symmetric ways using the above five tiles in mosaic designs.

3. Creating Decorative Pentagram and Decagram Using Tiling Techniques

Figure 6 illustrates two decorated decagrams. The left decagram is an actual tiling created by an anonymous artist. The right decagram has been created by the author. The two decagrams are covered by two different sets of motifs. Nevertheless, they have the following similarities:

(a) Both tessellations have 10-fold rotational symmetry, and (b) The vertices of the shapes in both tessellations are located at the center of the small decagrams that constitute the tiling.

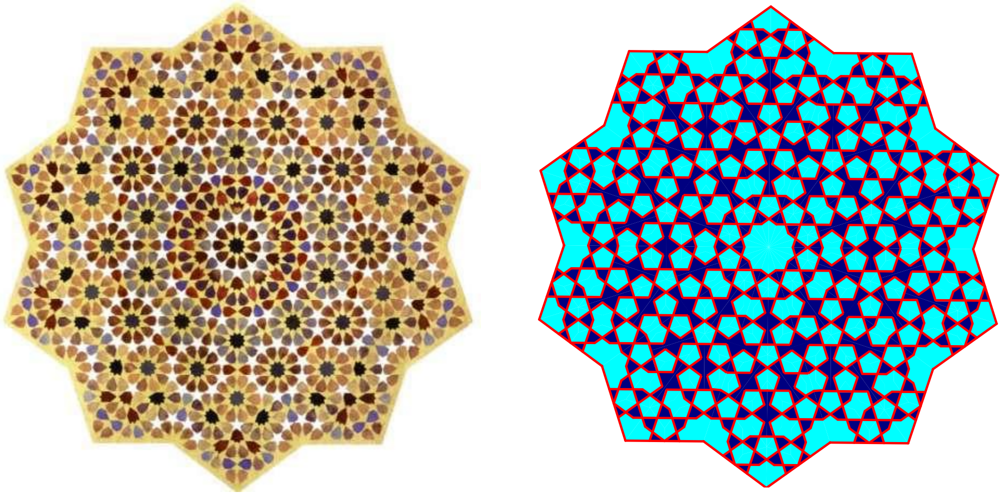
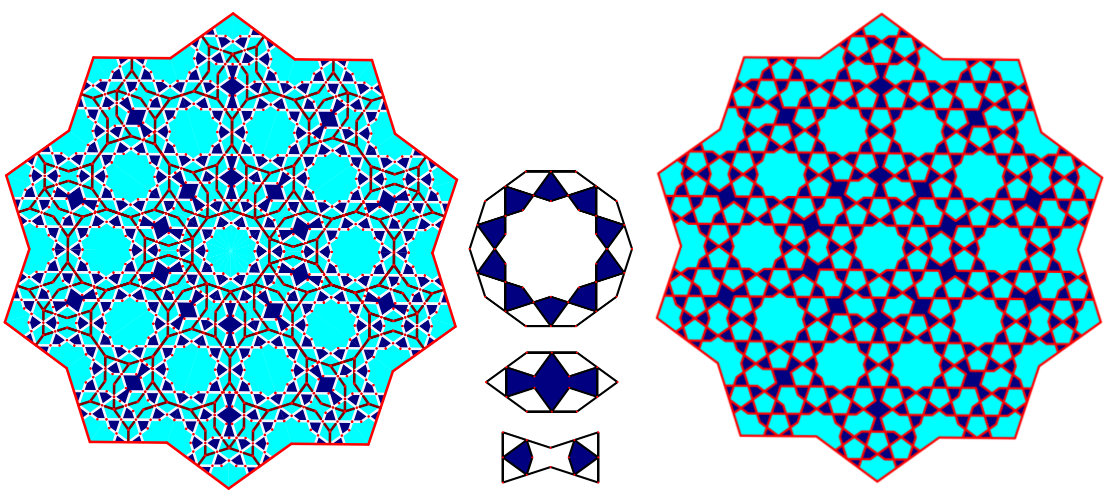


Figure 6: *An actual tiling decagram, and* a *tiling decagram created by the author.*

We notice that the convex corners of these two shapes that have 108º angles are covered by 3/10 portion of a decagram, and the concave corners with angles 216 º, are covered by 3/5 portion of a small decagram (It is a mathematical property regardless of the size of the small decagrams in any such a tiling).

One might ask whether there are tessellations other than the right image in Figure 6 that cover the interior of a decagram using the 5 sâzeh modules. To answer this question we notice that one way of constructing such a tiling is the use of girih tiles shown in 7(b) [7]. Figure 7(a) shows the tiling of the decagram when we use the girih tiles exhibited in Figure 7(b). Now if we disregard the boundary lines of the girih tiles, then we will come up with another tessellation as in Figure 7(c). This decagram tiling can be extracted from the tiling on the tympanum of Darb-Imam Shrine, Isfahan [2]. From these two different tilings we conclude that the tiling of a decagram, using the aforementioned constraints is not unique.



(a) (b) (c)

Figure 7: (a) *The construction of the tiling in Figure 6.b using girih tiles*, (b) *Three girih tiles*, (c) *The same tiling as in (a) which is now covered by the sâzeh tiles presented in Figure 5.*

Now we move our attention from tiling of a decagram to the tiling of a pentagram. Figure 8 presents a tiling for the pentagram using different motifs than the sâzeh modules. The left image in Figure 8 shows how such a tiling is possible using the compass-straightedge construction. The right image in Figure 8 shows the computer generated tiling that is the replica of an existing tessellation [8-9].

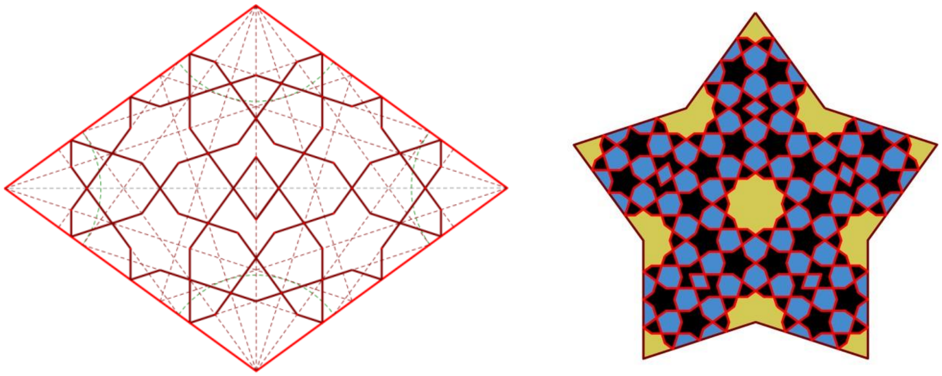
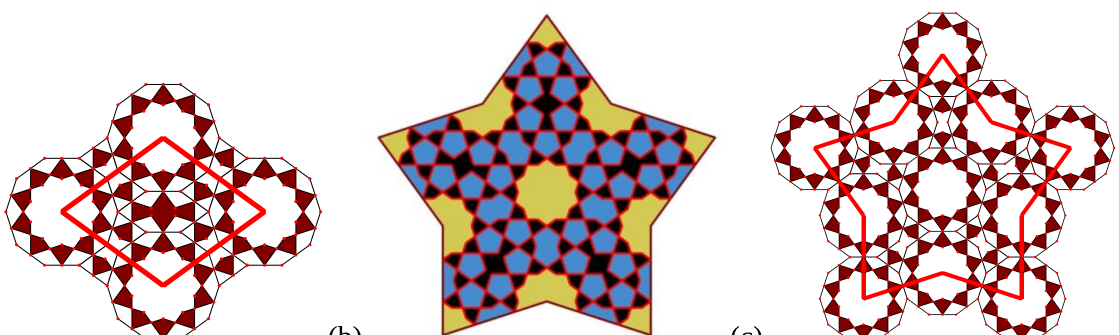


Figure 8: *The compass-straightedge construction of a tiling on a 72º rhombus, and the tiling of a pentagram using the set of motifs generated in this construction.*

Rather than the tremendously complicated task of constructing such a tiling for sâzeh motifs using the compass-straightedge construction similar to Figure 8, we use the girih tiles to create a region that includes a 72º rhombus as in Figure 9(a-b). Using this rhombus and a rotation of 72º about its 72º vertex will generate the desired pentagram tessellation. Figure 9(c) shows another tessellation for the pentagram to illustrate that the tiling is not unique.



(a) (b) (c)

Figure 9: (a) *An arrangement of girih tiles for generating a 72º rhombus*, (b) *The pentagram tiling*, (c) *Another pentagram tiling using a different girih tiles arrangement.*

4. The Art of Tazhib

In more elaborated structures such as buildings erected during the Safavid dynasty (1501- 1736 CE), the art of tiling was usually accompanied and integrated with the art of *Tazhib*. Tazhib is a Persian non-figurative illumination art form that has been traditionally used for the decoration of the title/subtitle pages as roundel (*Shamseh*), or the margins (*Hashiyeh*) of holy books and epic poems. Nevertheless, its elements have appeared in Persian rugs and mosaics as well. In most cases an artwork is created by repeating a fundamental region pattern called a *Vagireh*. The root of this art, which has a fundamental connection with vegetable and plant portrayals, can be found in pre-Islamic Persia, especially in the decorative art created during the *Sassanid* dynasty (224-651 CE). It is said that *Mani*, a prophet of Persian origin of 3rd century CE, illustrated his holy book of *Ergenk*, or *Estenk*, using Tazhib and Miniature paintings.

To understand the process thoroughly we follow the artworks of Mojgan Lisar, a tazhib artist from Enschede, the Netherlands. The components of tazhib motifs include abstract interlacing and chain spiral patterns that are called *Eslimi* patterns and stylized floral designs that are called *Khatâyi* patterns and *Khatâyi* flowers. To create a tazhib pattern, at first an artist makes a few stylized curves and spirals to determine and illustrate the boundaries of the artwork. This first set of curves is the Eslimi part of the work. The Eslimi part consists of tick branches without flowers but very stylized leaves (Figure 10).

Figure 10: *Some Eslimi leaves and branches.*

Then, the artist adds the Khatâyi curves and spirals that include many different sized flowers and leaves (Figure 11). The artist may add more details to complete the design. The branches in the Khatâyi part are generally thinner than the Eslimi branches. The larger Khatâyi flowers with more complex structures have their own specific names such as *Lâleh Abâsi* and *Gol Panbehyi* flowers.

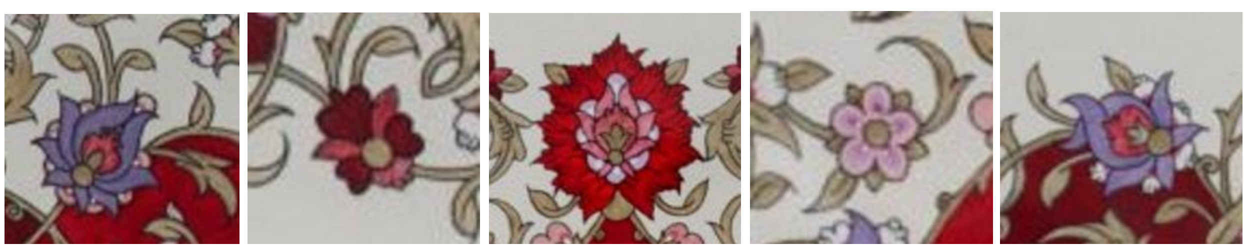


Figure 11: *Some khatâyi flowers with different number of petals and sizes.*

In a few occasions the artist may include emblems and abstract animal designs such as deer, canaries, and nightingales. The list of possible components and their varieties is long and beyond the focus of this article. Interested readers are referred to resources such as [1], [5], and [6].

5. The Decagram and the Art of Tazhib

Using the two concentric congruent regular pentagons presented in Figure 2, Mojgan Lisar constructed the decagram. A fundamental region for creating the entire tazhib design is a 36° sector of this decagram. After designing this sector and transferring it to transparency paper, the artist covered the entire interior of the decagram using a rotation of 36°. In some cases an artist may decide to use one half of the fundamental region only, a sector of 18°, for the original design and constructs the other half using a reflection, as is in this artwork.

To complete the coloring of this tazhib decagram, the artist used a series of blue colors only. In traditional artwork an artist combines hot colors (such as red, brown, and so on), with cold colors (blue, purple, and so on). Nevertheless, to harmonize the work in a way that the eye of the beholder becomes more attracted to fine details and the viewer feels more at peace and harmony with the piece, the artist used blue color combinations only. This way the viewer's concentration is not broken by the color contrast. Figure 12 exhibits the final artwork and all the used blue colors.

Figure 12: *From top left to bottom right – the process of designing and painting the tazhib decagram.*

6. An Artistic Collaboration Combining the Two Art Forms of Tiling and Tazhib as the Conclusion

Figure 13, the *Blue Sun,* is a collage of the two aforementioned works of art of tiling and tazhib. The front image is the created decagram *shamseh* (the roundel tazhib) as in the previous section. The mosaic design on the back is a two-level self-similar tiling that has been made based on the decagram presented in Figure 7.b. The tiling in the center of the shamseh is a smaller version of the same decagram as in the back.

Note that all the tiling images and their arts have been created by the author. All the tazhib artworks and sketches have been created by the tazhib artist.

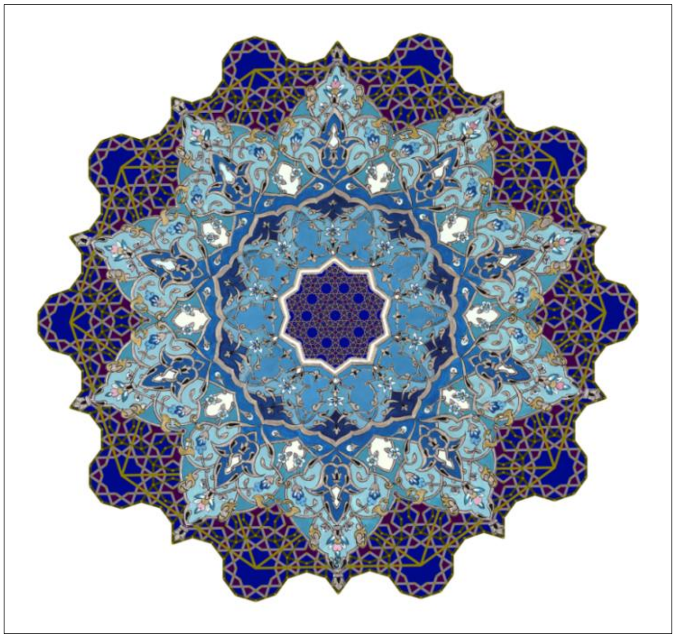


Figure 13: *Blue Sun*

This joint artwork has been exhibited in the *Joint Mathematics Meeting* Mathematical Art Exhibition in San Diego [3], and the *2013 Bridges Enschede* in the Netherlands [4].

References

[1] Agha Miri, A.H., *Designing Tazhib*, Yassavoli Publications, 2008, Tehran, Iran.

[2] Bonner, J., *Three traditions of self-similarity in fourteenth and fifteenth century Islamic geometric ornament,* Proceedings of the ISMA/Bridges: Mathematical Connections in Art, Music, and Science, R. Sarhangi and N. Friedman, eds, Granada, Spain, 2003, pp. 1-12.

[3] Fathauer R. and Nathan S Selikoff, eds, *Joint Mathematics Meeting Art Exhibition Catalog* 2013, San Diego, California, USA.

[4] Fathauer R. and Nathan S Selikoff, eds, *Bridges International Conference Art Exhibition Catalog* 2013, Enschede, the Netherlands.

[5] Honarvar, M.R., *Symbols of Iranian Illumination and Carpet Designing*, Yassavoli Publications, 2011, Tehran, Iran.

[6] Honarvar, M.R., *The Illumination Models, Eslimi and Khataei Decorative Designs*, Yassavoli Publications, 2008, Tehran, Iran.

[7] Lu, P.J. and P.J. Steinhardt, *Decagonal and quasi-crystalline tilings in medieval Islamic architecture*, Science 315, 2007, pp. 1106 – 1110.

[8] Maheronnaqsh, M., *Design and Execution in Persian Ceramics*, Reza Abbasi Museum Press, 1984, Tehran, Iran.

[9] Sarhangi, R., *Interlocking Star Polygons in Persian Architecture: The Special Case of the Decagram in Mosaic Designs*, Nexus Network Journal, Volume 14, N. 2, 2012, pp. 345-372.