The Art of Geometry

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Abstract

This paper deals with new methods for modelling and studying shape in mathematics. It aims to show similarities between mathematical and artistic solutions applied in the creation of a piece of work – artistic or mathematical. Minkowski operations on point sets are introduced to create complex forms as point set combinations and used as generating principles for modelling various interesting geometric structures such as point mosaics, flexible curves and smooth surface patches in Euclidean space.

Introduction

Various mathematical disciplines, including branches of geometry, take distinct and helpful perspectives in their approach to the study of the shapes of geometric structures. Differential geometry investigates properties of curvature and local behaviour. Topology focuses on dimension and outer structures.. Coordinate geometry uses analytic representation to determine position and shape of geometric figures with respect to a fixed coordinate system. The visualization of these geometric figures from various viewpoints is important in descriptive geometry and projection methods. Geometric modeling attempts to generate and modify particular shapes and to determine different forms of the representation for each.

Consider the representations of the most basic geometric figure: a point. A point can be determined synthetically, by simply drawing a dot in space, or analytically, by referring to its particular position with respect to certain fixed coordinate system. In algebra, a point can be represented by its position vector from the associated vector field, or it can be regarded as an element of a certain set of points, discrete and finite or continuous and infinite, which is usually a well-defined algebraic structure.

Geometric figures of dimension greater than zero, including curves, surfaces and solids, can be described as special configurations of points. These point configurations form various geometric structures, which often demonstrate interesting features of symmetry, repeating motifs arranged in a mosaic, or certain patterns repeatedly appearing in various transformed positions and formations. Configurations can be generated randomly, but usually there exists a special algebraic tool suitable for their modification.

Minkowski Sum and Minkowski Product

The Minkowski point set operations, Minkowski sum $\oplus$ and Minkowski product $\otimes$, are abstract algebraic operations defined on point sets in any geometric space, which can serve as powerful tools for modelling unusual geometric forms with added aesthetic value.

Let A and B be nonempty point sets in the n-dimensional Euclidean space $\mathbf{E}^n$ with orthogonal Cartesian coordinate system $\langle O, x_1, x_2, \dots, x_n\rangle$ associated with the vector space $V(\mathbf{E}^n)$ with orthonormal basis $\langle O, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\rangle$. The relationship between the two structures can be represented as an isomorphism $\pi : \mathbf{E}^n \to V(\mathbf{E}^n)$, by which any point $M \in \mathbf{E}^n$ with Cartesian coordinates $m = [x_1^m, x_2^m, \dots, x_n^m]$ is attached its position vector $\pi(m) = \mathbf{m} = x_1^m\mathbf{e}_1 + x_2^m\mathbf{e}_2 + \dots + x_n^m\mathbf{e}_n = (x_1^m, x_2^m, \dots, x_n^m) \in V(\mathbf{E}^n)$.

The sum of two points a and b in the space $\mathbf{E}^n$ is the point $c \in \mathbf{E}^n$, whose position vector from the associated vector field $V(\mathbf{E}^n)$ is the sum of the position vectors of points a and b:

$$a, b \in E^n,\ \pi(a) = \mathbf{a},\ \pi(b) = \mathbf{b} \Rightarrow a + b = c \in E^n \Leftrightarrow \pi(c) = \mathbf{c} = \mathbf{a} + \mathbf{b}.$$

This isomorphism forms a background for the definition of algebraic operations between points in the space $\mathbf{E}^n$. Consequently, using these operations between points, we may define more complex operations of sum and product of two point sets leading to the abstract algebraic concepts of Minkowski sum and Minkowski product of point sets and Minkowski set operators.

Definition 1. The Minkowski sum of the point sets A and B in $\mathbf{E}^n$ is a point set in $\mathbf{E}^n$ whose points are sums of all points from set A with all points from set B, therefore

$$A \oplus B = \{a + b \mid a \in A, b \in B\},\ \ A \oplus \varnothing = \varnothing.$$

Alternatively, we can write

$$A \oplus B = \bigcup_{b \in B} A^b,$$

where A^b is the set generated as a translation of the set A by the vector $\mathbf{b} \in V(B)$ attached as position vector to a fixed point $b \in B$, that is

$$A^b = \{m \in \mathbf{E}^n \mid \mathbf{m} = \mathbf{a} + \mathbf{b}, \mathbf{a} \in V(A)\} = \{a + b \mid a \in A\}.$$

We define the product of two points a and b in the space $\mathbf{E}^n$ to be the point $c \in \mathbf{E}^d$, whose position vector from the associated vector field $V(\mathbf{E}^d)$ is the outer (wedge) product of position vectors of points a and b

$$a, b \in \mathbf{E}^n,\ \pi(a) = \mathbf{a},\ \pi(b) = \mathbf{b} \Rightarrow a \wedge b = c \in \mathbf{E}^d \Leftrightarrow \pi(c) = \mathbf{c} = \mathbf{a} \wedge \mathbf{b},$$

where $d = n(n-1)/2$.

Definition 2. The Minkowski product of two point sets A and B in the space $\mathbf{E}^n$ is a point set in $\mathbf{E}^d$, whose points are the outer products of all points from set A with all points from set B:

$$A \otimes B = \{a \wedge b \mid a \in A, b \in B\},\ \ A \otimes \varnothing = \varnothing.$$

Recall that the operation of outer (wedge) product of two vectors satisfies the following rules:

1. $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$
2. $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$
3. $k(\mathbf{a} \wedge \mathbf{b}) + l(\mathbf{a} \wedge \mathbf{b}) = (k + l)(\mathbf{a} \wedge \mathbf{b})$
4. $(k\mathbf{a} + l\mathbf{b}) \wedge (m\mathbf{a} + n\mathbf{b}) = (kn - ln)(\mathbf{a} \wedge \mathbf{b})$
5. $\|\mathbf{e}_i \wedge \mathbf{e}_j\| = 1,\ i, j = 1, 2, \ldots n,\ i \neq j.$

The outer product of the two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ from the vector space $V(\mathbf{E}^2)$ is the vector $\mathbf{a} \wedge \mathbf{b}$ in the one dimensional vector space $V(\wedge^2(\mathbf{E}^2)) = V(\mathbf{E}^1)$, which can be calculated using the above rules

$$\mathbf{a} \wedge \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = (a_1 b_2 - a_2 b_1)(\mathbf{e}_1 \wedge \mathbf{e}_2),$$

where the resulting vector $\mathbf{e}_1 \wedge \mathbf{e}_2$ is outside the original vector space $V(\mathbf{E}^2)$. The vector $\mathbf{e}_1 \wedge \mathbf{e}_2$ is a unit vector forming the basis of the vector space $V(\mathbf{E}^1)$ that can be represented as the orthogonal complement of vector space $V(\mathbf{E}^2)$ in the covering vector space $V(\mathbf{E}^3)$ that is formed as the linear sum of the 2-dimensional vector space $V(\mathbf{E}^2)$ and the one-dimensional vector space $V(\mathbf{E}^1)$.

The outer product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ from the vector space $V(\mathbf{E}^3)$ is the vector $\mathbf{a} \wedge \mathbf{b}$ from the same vector space calculated as

$$\mathbf{a} \wedge \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) =$$
$$= (a_1b_2 - a_2b_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) + (a_1b_3 - a_3b_1)(\mathbf{e}_1 \wedge \mathbf{e}_3) + (a_2b_3 - a_3b_2)(\mathbf{e}_2 \wedge \mathbf{e}_3),$$

where vectors $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_1 \wedge \mathbf{e}_3 = -\mathbf{e}_2$ and $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1$ form the basis of space $V(\mathbf{E}^3)$. Therefore, in the 3-dimensional vector space, the outer product of two vectors is the usual cross-product.

Based on the scalar multiple of a vector, we can define a scalar multiple of a point, and consequently of a point set.

Definition 3. The multiple of a point set $\boldsymbol{A}$ in the space $\mathbf{E}^n$ by the scalar $k \in R$ is the point set $A_k \subset \boldsymbol{E}^n$ of all such points, whose position vectors are k-multiples of the position vectors of all points from the set A, hence

$$A_k = k \cdot A = \{km \mid m \in A, m \mapsto \mathbf{m} \Rightarrow km \mapsto k\mathbf{m}\}, \; k \in R.$$

The set A_k is also denoted as a dilatation of set A by a non-zero scalar $k \in R$, and it can be interpreted geometrically as a homothetically scaled image of the point set A in the homothety with center at the origin, O, with scaling coefficient k. Any symmetric configuration of points in the set A is therefore an invariant property that appears also in the configuration of points in the k-multiple image - the point set A_k.

A generalization of these basic operations on point sets leads to the abstract algebraic concepts of Minkowski set combinations determined as results of Minkowski set operators defined on point sets. Three different forms can be determined in general, according to variations of the two operations of Minkowski sum and Minkowski product, namely Minkowski linear combination, Minkowski product combination, and Minkowski mixed combination.

Minkowski Linear Combinations of Point Sets

Consider the arbitrary point sets A and B in the space $\mathbf{E}^n$. Their Minkowski linear combination is the set C in the space $\mathbf{E}^n$ generated as Minkowski sum of the k-multiple of set A and the l-multiple of set B

$$C = k \cdot A \oplus l \cdot B = A_k \oplus B_l = \{ka + lb \mid a \in A, b \in B\}, \;\; k, l \in R.$$

Several illustrations of Minkowski linear combinations of two discrete point sets are presented in Figure 1. Symmetric arrangements of point configurations in the basic sets A and B result in inherited symmetric forms of their Minkowski sum, while Minkowski linear combinations demonstrate flexible dynamic mosaics that can be re-arranged by choosing particular values of multipliers k and l of the respective Minkowski linear combinations of point sets A and B.

A one-dimensional geometric figure, a curve segment, is an infinite set of points that can be represented analytically in the vector form by a vector function of one real variable on a given interval. Taking two different curve segments, we can parameterize them over the same interval using the same single variable, $t \in I \subset \boldsymbol{R}$. Let us denote such curves as equally parameterized on the interval I.

The Minkowski linear combination of these equally parameterized planar curve segments, $\mathbf{r}(t)$ and $\mathbf{s}(t)$, is a curve segment equally parameterized on the same interval I, while its vector representation is the linear combination of their vector maps

$$\mathbf{p}(t) = k\mathbf{r}(t) + l\mathbf{s}(t), \; t \in I,$$

for arbitrary real values of coefficients k and l.

The dynamic configuration defined by the Minkowski linear combination enables a flexible smooth change of the resulting curve shape with respect to the values of multipliers k and l in particular combinations, as illustrated in the examples presented in Figure 2.

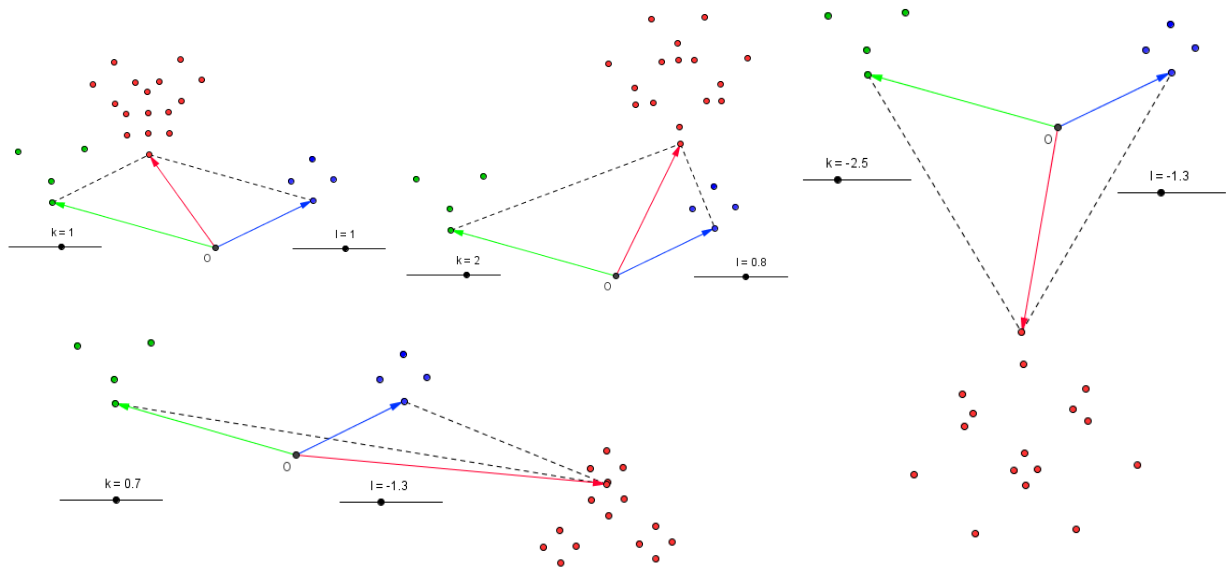

Figure 1: *Minkowski linear combinations of two discrete point sets*.

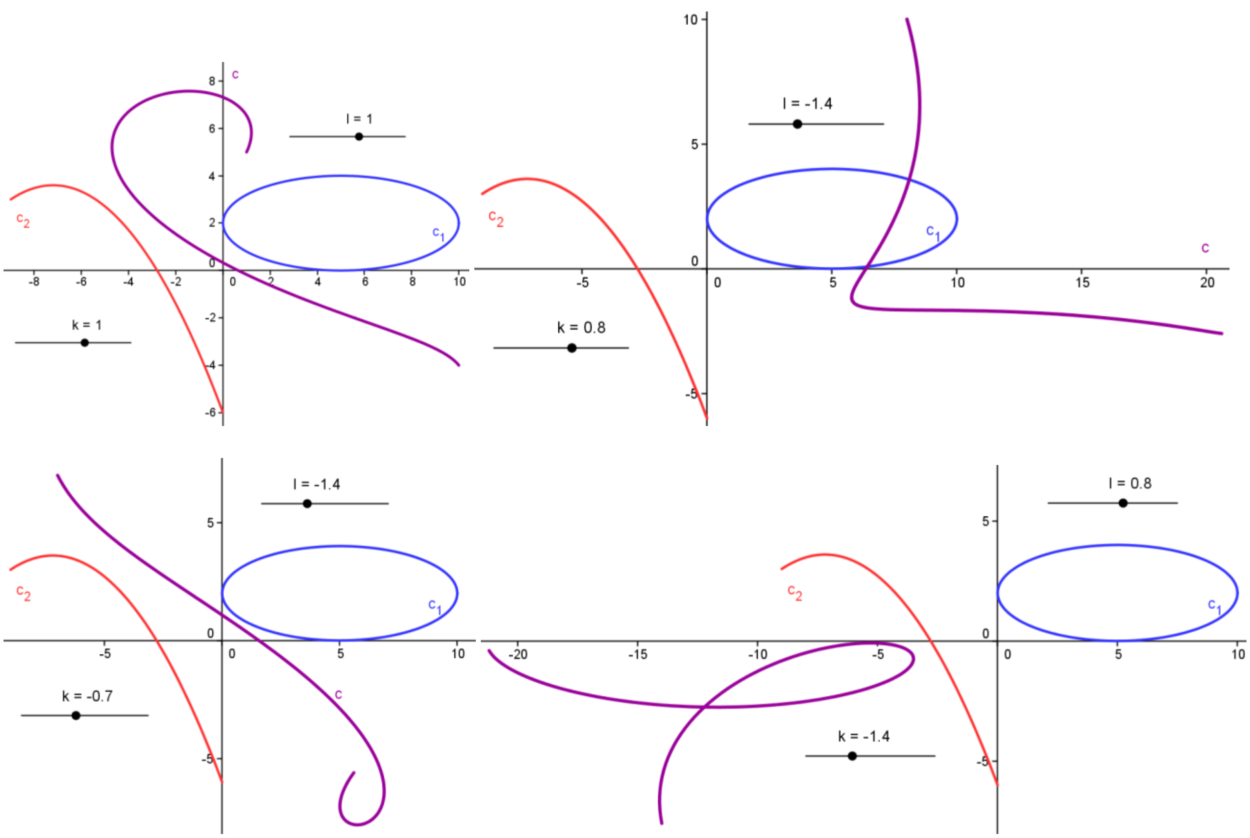

Figure 2: *Minkowski linear combinations of ellipse and parabolic segment.*

The Minkowski linear combinations of curve segments with different parameters, $u \in I \subset \boldsymbol{R}$ and $v \in K \subset \boldsymbol{R}$, determine surface patches that are analytically represented by vector functions in two variables, $\mathbf{p}(u, v) = k\mathbf{r}(u) + l\mathbf{s}(v)$, $(u, v) \in I \times K \subset \boldsymbol{R}^2$, for details see [1], [2], and [3]. Minkowski linear combinations can be regarded as a modelling tool for shaping various interesting forms of surface patches as two-dimensional figures in the 3-dimensional space $\mathbf{E}^3$; many interesting examples can be found in [4] and [5]. The surface patches illustrated in Figure 3 are Minkowski linear combinations of a parabolic arc and the lemniscate of Bernoulli, two planar curve segments positioned in two perpendicular planes in 3-dimensional space.

The Minkowski linear combination of two curve segments embedded in higher dimensional spaces are well-defined and can model geometric figures in higher dimensions, see [6] and [7]. Visualisation of these geometric structures can lead to views of surface patches demonstrating certain unusual artistic and aesthetic values and can be regarded as objects that stimulate our brains to develop higher dimensional imagination. Orthographic projection from 4D to different 3D subspaces can be easily obtained, and some examples are given in Figure 4.

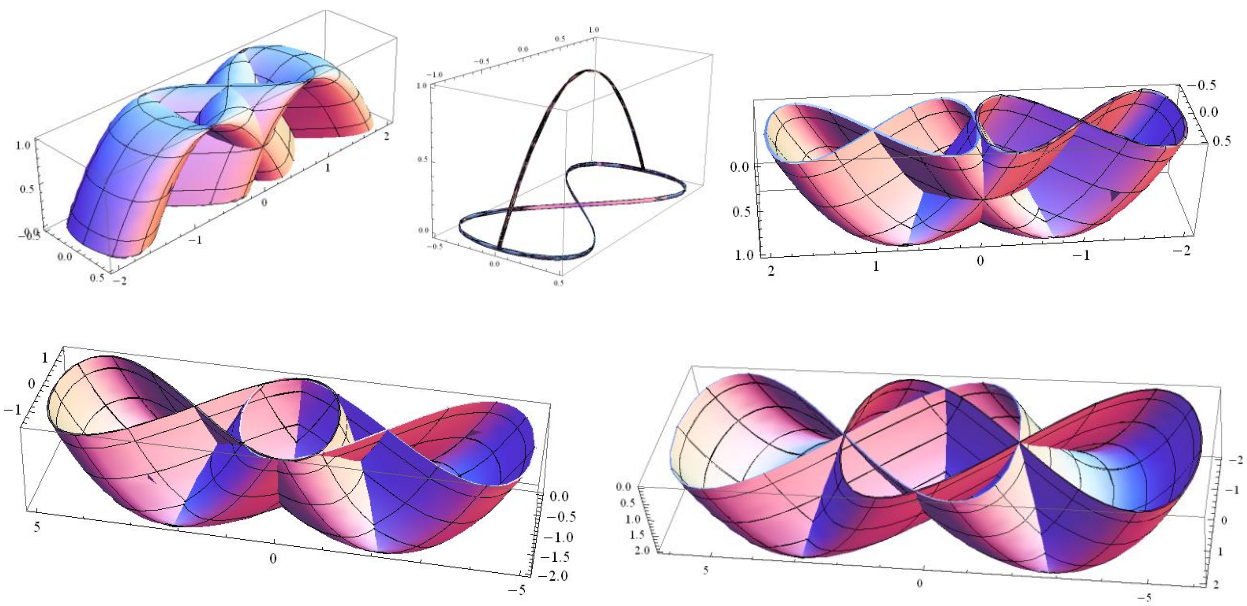

Figure 3: *Minkowski linear combinations of two curve segments in 3D.*

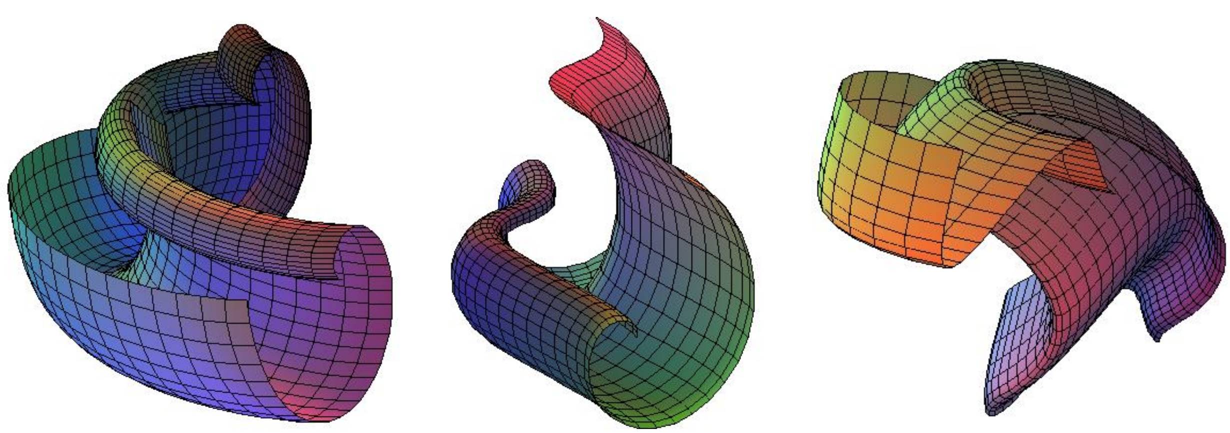

Figure 4: *Orthographic views of the Minkowski linear combinations of two helical segments in 4D.*

Minkowski Product Combinations of Point Sets

The Minkowski product combination of two point sets A and B in the space $\boldsymbol{E}^n$ is the point set C in the space $\boldsymbol{E}^n$ defined as the Minkowski product of the k-multiple of set A and the l-multiple of set B

$$C = k \cdot A \otimes l \cdot B = A_k \otimes B_l = \{ka \wedge lb \mid a \in A, b \in B\}, \ \ k, l \in R \, .$$

The Minkowski product combination of two equally parameterized curve segments is a curve segment. The example in Figure 5 shows the Minkowski product and Minkowski sum compared to the Minkowski linear and Minkowski product combinations of the circle and shamrock curve located in perpendicular planes. The illustrations in Figure 6 are views of the Minkowski product of curves from Figure 3. The Minkowski product of the shamrock curve and versiére located in different 3-dimensional subspaces of $\mathbf{E}^4$ determine a spectacular surface patch presented in different views in Figure 7.

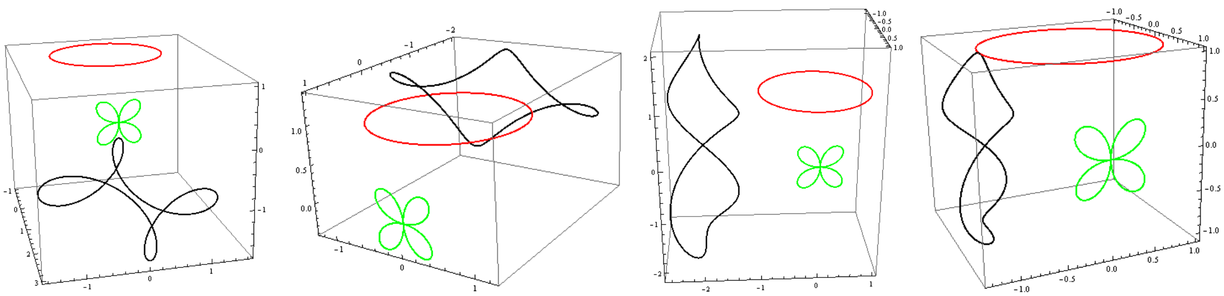

Figure 5: *The Minkowski sum, linear combination, product and product combination.*

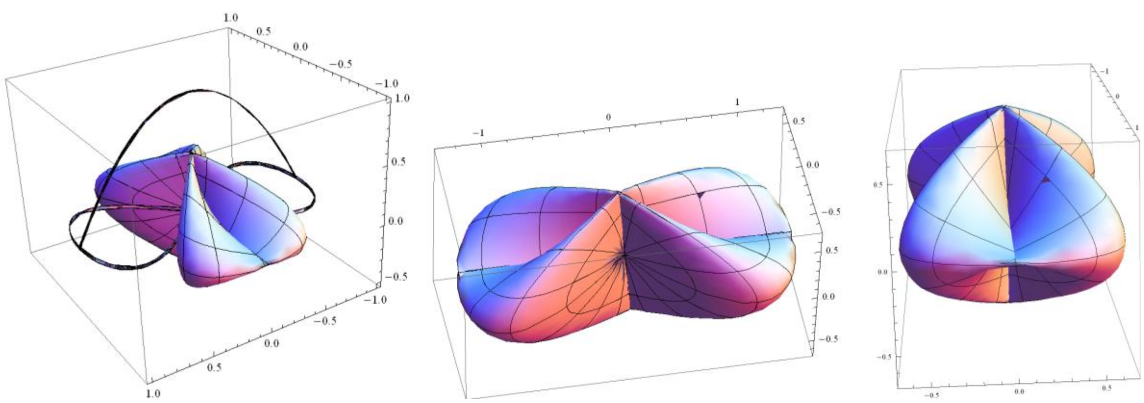

Figure 6: *Views of the Minkowski product of two curve segments in 3D.*

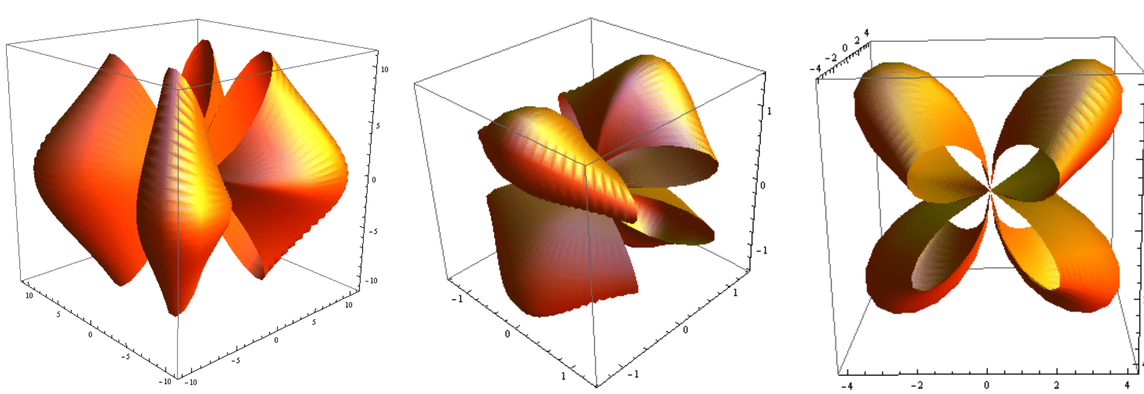

Figure 7: *Three different views of the Minkowski product of the shamrock curve and versiére.*

Minkowski Mixed Combinations of Point Sets

The concept of the Minkowski mixed combination of three point sets can be introduced based on the Minkowski sum and Minkowski product of two point sets.

Definition 4. Let A, B, and C be point sets in the space $\boldsymbol{E}^n$ and k, l, and h be real numbers. The Minkowski mixed combination of point sets A, B, and C is the point set W in the space $\boldsymbol{E}^d$ defined by the following formula

$$W = (k \cdot A \oplus l \cdot B) \otimes h \cdot C = (A_k \oplus B_l) \otimes C_h =$$
$$= \{(ka + lb) \wedge hC \mid a \in A, b \in B, c \in C\}.$$

The Minkowski mixed combination of three equally parameterized curve segments is a curve segment, while the combination of three differently parameterized curves segments results in a 3-dimensional solid. The surface patch can be determined as a Minkowski mixed combination of two equally parameterized curve segments $\mathbf{r}(u)$ and $\mathbf{s}(u)$, $u \in I$, with the third one, which is differently parameterized $\mathbf{p}(v)$, $v \in K$. Therefore, we can define different combinations and investigate the form of the resulting figures, which include

$${}^{1}\mathbf{r}(u, v) = (k\mathbf{r}(u) + l\mathbf{s}(u)) \wedge h\mathbf{p}(v),\ (u, v) \in I \times K, \text{ and}$$

$${}^{2}\mathbf{r}(u, v) = (k\mathbf{r}(u) + l\mathbf{p}(v)) \wedge h\mathbf{s}(u),\ (u, v) \in I \times K.$$

Three circles located in perpendicular planes generate a planar curve as their Minkowski mixed product, which is illustrated in Figure 8 on the left, while the surface patch generated as the Minkowski mixed product of the same 3 circles with 2 of them equally parameterized, is illustrated in Figure 8 on the right. Views of some Minkowski mixed combinations of these three circles are illustrated in Figure 9.

Interesting shapes and forms of generated surfaces can be used for purposes of graphic design, in visualisations or in morphing. The underlying abstract principles of Minkowski sum and Minkowski product defined on point sets in some geometric space provide a tool for the generation of synthetic visual and analytic representations. Thus the intrinsic geometric properties of the created objects can be studied using the methods of differential geometry, and the specific properties inherited from the operand sets can be detected and defined. The new objects are therefore interesting from both the aesthetic and the theoretical mathematical point of view.

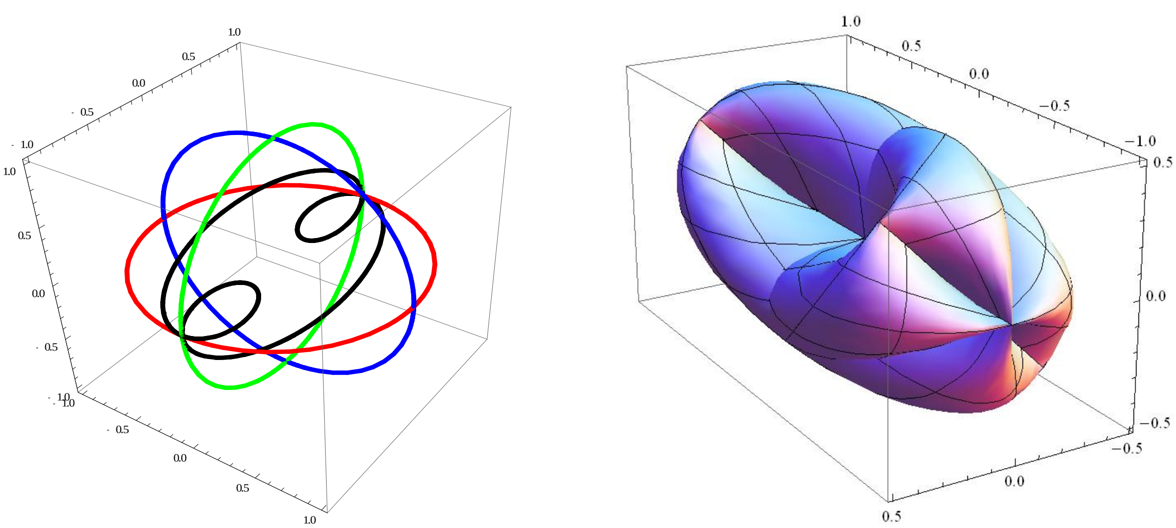

Figure 8: *The Minkowski mixed product of three circles.*

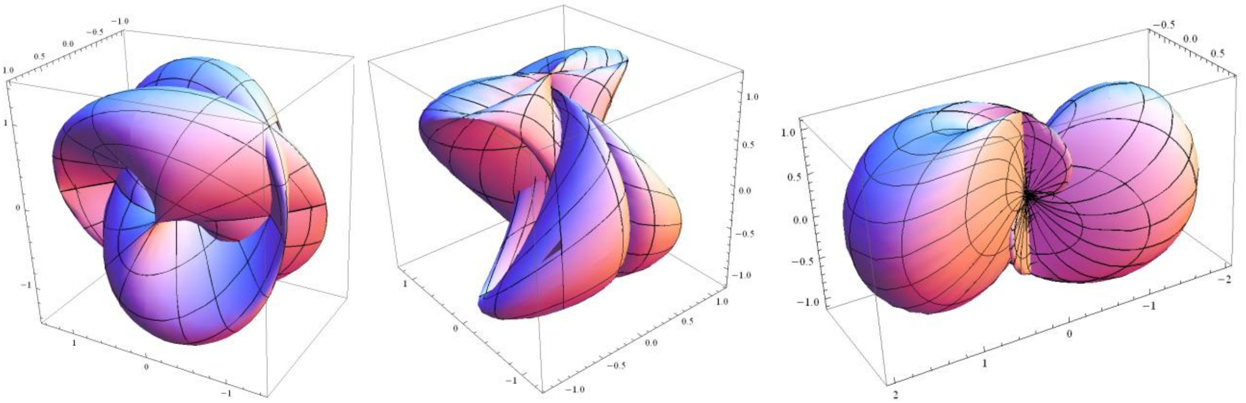

Figure 9: *Minkowski mixed combinations of three curve segments in 3D.*

Conclusions

The abstract algebraic operations of the Minkowski sum and the Minkowski product serve as basic laws for the modelling of point set combinations, and they represent powerful tools for design and smooth dynamic modification of variable geometric configurations. They are useful for multiple purposes, such as

- morphing in computer graphics
- modelling of smooth manifolds in 2, 3 and higher dimensional spaces
- creative design of unusual forms of point sets in $\mathbf{E}^n$

Generated geometric structures demonstrate specific geometric properties determined by their generating principles and based on properties of original operand point sets of these abstract algebraic set operations. Mathematical and visual construction plans are built on similar principles and realise the beauty and elegance of the object in the sense of its flawlessness, uniqueness and inner determining laws.

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