Artfully Folding Hexagons, Dodecagons, and Dodecagrams

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Abstract

Folding dissections are introduced for hexagons, dodecagons, and dodecagrams. Each folding dissection transforms one of these figures to a similar figure but of a different height. The goal is to minimize the number of pieces in the folding dissection, while at the same time exploiting symmetry to create beautiful objects that fold magically before our eyes. For regular hexagons, the dissections transform a regular hexagon of height h to a regular hexagon of height $n * h$, where n is, in turn, 3 or 4 or 9 or 16 or 25. For regular dodecagons, our dissection transforms one dodecagon to another twice as high. For the 12-pointed star $\{12/2\}$, we give a dissection to a star 3 times as high, and also one to a star twice as high. The design of these various folding dissections is explored.

1 Introduction

Geometric dissection is the art of cutting of one or more geometric figures into pieces that we can rearrange to form other geometric figures [1]. One of the earliest examples is the elegant 4-piece dissection of two congruent squares to a larger square, discussed more than two millennia ago by Plato in his *Menon* and his *Timaeus*. Just cut each of the two small squares into two congruent isosceles right triangles and then assemble these four pieces to form the larger square. Indeed, we can hinge the pieces from each small square, so that they swing around in the plane and then slide together to form the larger square, as discussed in [2].

It is also possible to adapt a dissection of two squares to one to make a *folding dissection*. Cut the squares into pieces which we then connect into one assemblage with "piano hinges," or tape, so that a square that is one level thick folds to a square that is uniformly two levels thick. This seems to require five pieces, as shown in Figure 1 and described in a paper by Lyle Pagnucco and Jim Hirstein [8]. The underlying idea had appeared already more than a century ago in Tandalam Sundara Rao's book on paper folding [9].

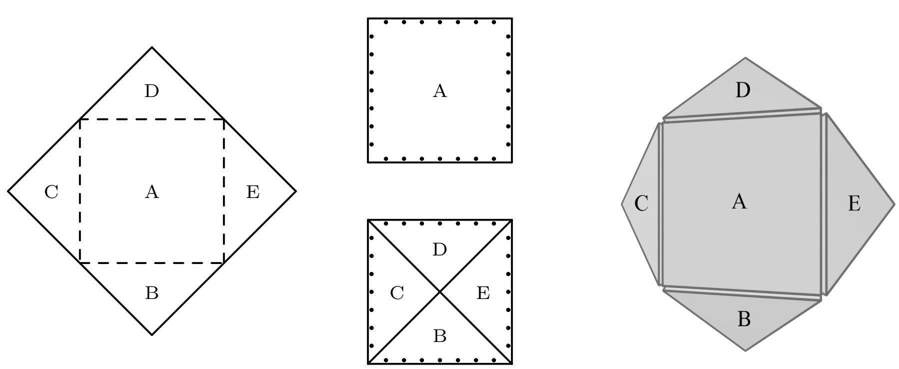

Figure 1 : *Folding dissection of a 1-level square to a 2-level square*

In Figure 1, a coarsely dashed line denotes a fold (or piano hinge) between two pieces that are adjacent on the same level, with the tape on the bottom side of the 1-high square. When a piece on one level is hinged to a piece on the level either above it or below it, the hinge will be denoted by a row of dots next to the shared edge on each level. We require each level to be the same positive thickness. Pieces may be on more than one level, but the portion of a piece on any level must be the union of one or more prisms, each as thick as a level. Here the focus is on attractive shapes and fundamental motion (rotations).

This article explores the following question here: Given two figures of equal volume, where one is p levels thick and the other is $q \neq p$ levels thick, and p and q are whole numbers, can we create just one assemblage that will fold to form each of the figures in turn? In such a case, our goals will include finding an assemblage with as few pieces as possible, finding a dissection with attractive symmetry (such as rotational or bilateral), and finding a dissection that is challenging to fold. Thus this activity falls at the juncture of kinetic art, mathematical optimization, and mechanical puzzles.

One application is for a tall, narrow object that might fit into a space rocket, that when launched into space would fold out into a squat, wide object like a dish antenna. Another application is for an aesthetically pleasing shape, with a cross section that is a star, that neatly folds from something taller to something shorter. In this paper we will limit the objects to be 6-pointed and 12-pointed polygons and stars, where the cross section of the transformed object is geometrically similar to the cross section of the original object.

2 Folding dissections of 1-level regular hexagons to multi-level regular hexagons

Since there is an elegant standard dissection of three hexagons to one, we reasonably expect to get a folding dissection of a 1-level hexagon to a 3-level hexagon. A 6-piece dissection from [7] does not seem to adapt well, but we can adapt a 7-piece dissection, as in Figure 2. Finely dashed lines in the 1-high hexagon represent tape on the top side, and coarsely dashed lines represent tape on the bottom side.

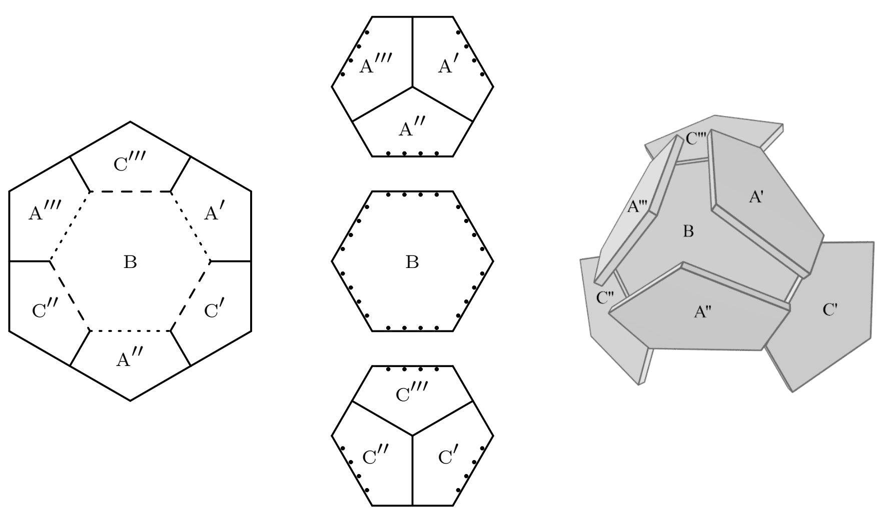

Figure 2 : *Folding dissection of a 1-level hexagon to a 3-level hexagon*

For a 1-level hexagon to a 4-level hexagon, it is not possible to use fewer than six pieces, because the cross-section of a 4-level hexagon cannot span more than one sixth of the perimeter of an equivalent 1-level hexagon. Thus the 6-piece dissection in Figure 3 is the best possible, in terms of fewest number of pieces.

There's more challenge with a 1-level hexagon to a 9-level hexagon. We could pack the 1-level hexagon with six copies of a small hexagon, and fill in each of the three corners with two half-hexagons, as in the 12-piece unhinged dissection that Ernest Freese described [6]. Yet placing the folds and cuts appropriately seems difficult, unless we cut three more of the hexagons in half. This latter approach produces the 15-piece

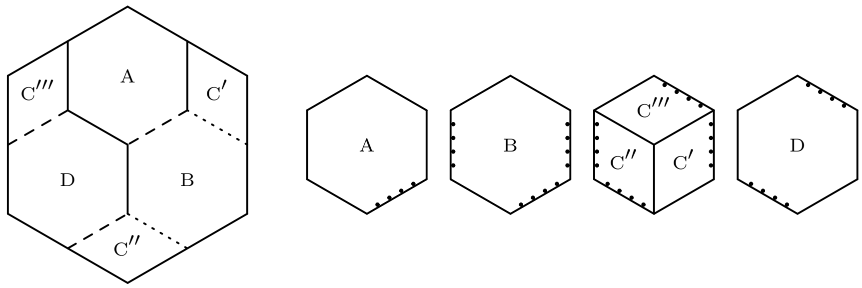

Figure 3 : *Folding dissection of a 1-level hexagon to a 4-level hexagon*

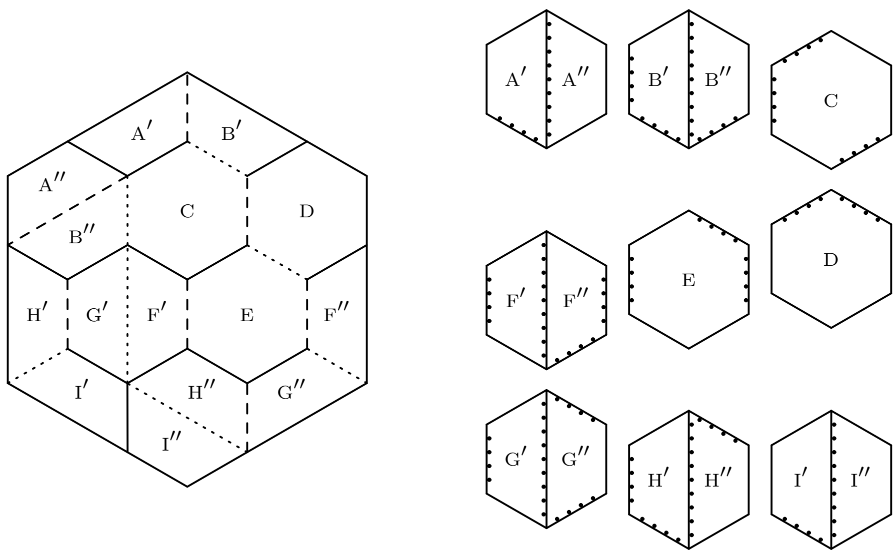

Figure 4 : *Folding dissection of a 1-level hexagon to a 9-level hexagon*

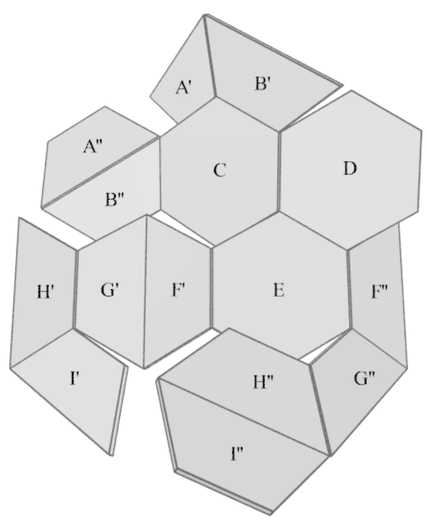

Figure 5 : *Perspective of a folding assemblage of a 1-level to 9-level hexagon*

folding dissection in Figure 4. Stacking the half-hexagons on top of each other in the 9-level hexagon seems to be a necessary trick. We display a perspective view in Figure 5.

The challenge increases with a 1-level hexagon to a 16-level hexagon. We choose a layout of pieces in the 1-level hexagon with 3-fold rotational symmetry, and then identify a way to fold so that, starting in the center, the hinged pieces wind around that central hexagon (piece *A*). This gives the 25-piece folding dissection in Figure 6. Figure 7 shows the nifty model that I constructed out of cherry wood, with the 1-level hexagon on the left and a partially folded-up configuration on the right.

The final hexagon example is of a 1-level hexagon to a 25-level hexagon. Once again we choose a layout of pieces in the 1-level hexagon with 3-fold rotational symmetry. We then identify a way to fold that

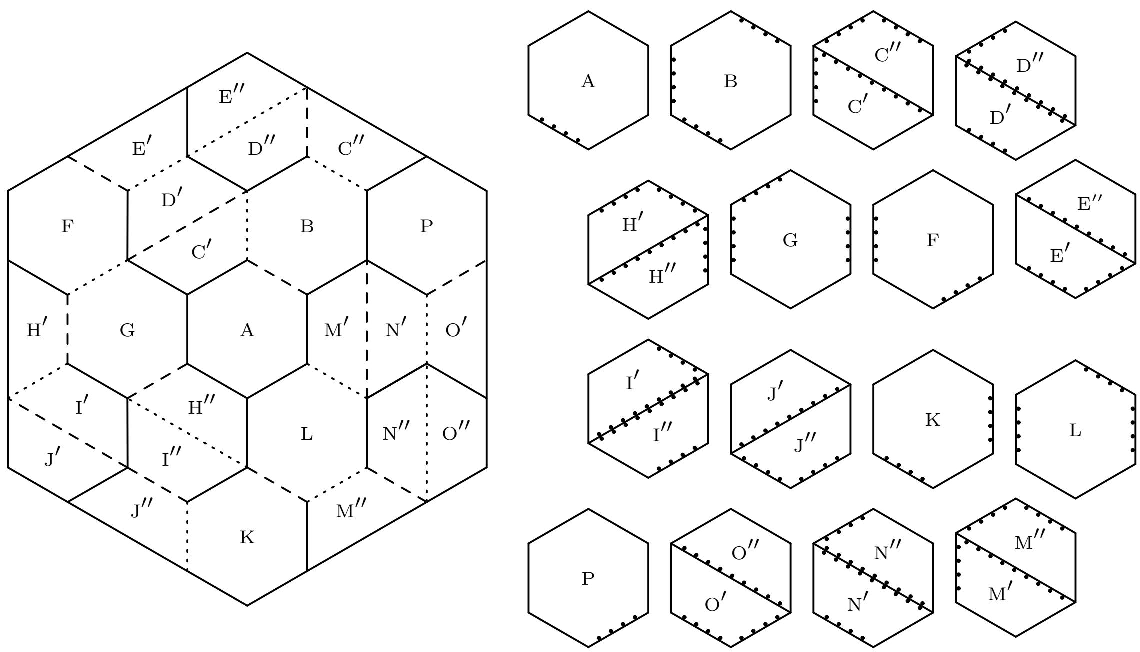

Figure 6 : *Folding dissection of a 1-level hexagon to a 16-level hexagon*

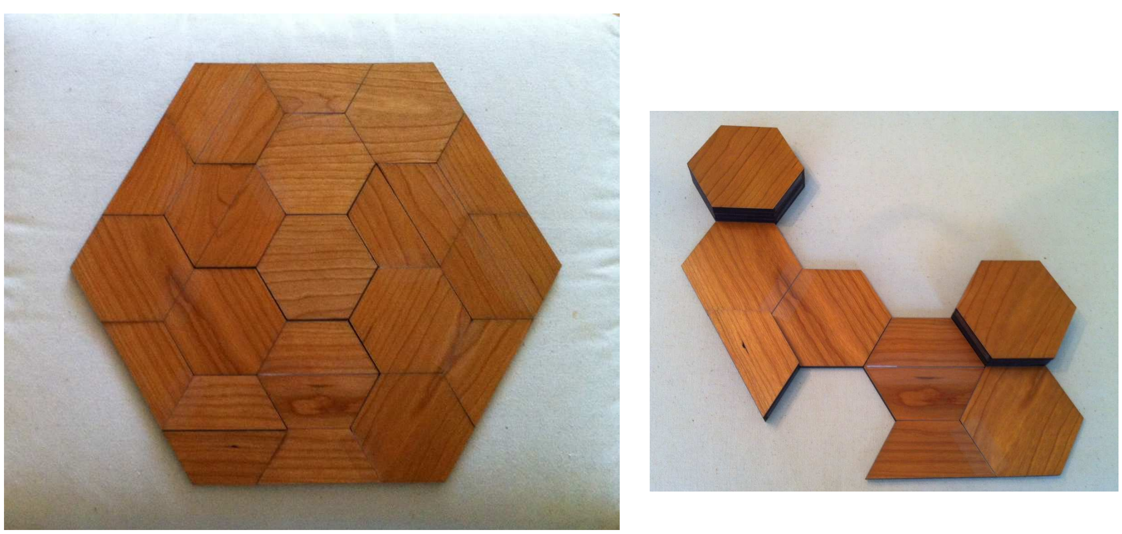

Figure 7 : *Folding model of a 1-level to 16-level hexagon: 1-level versus partially folded*

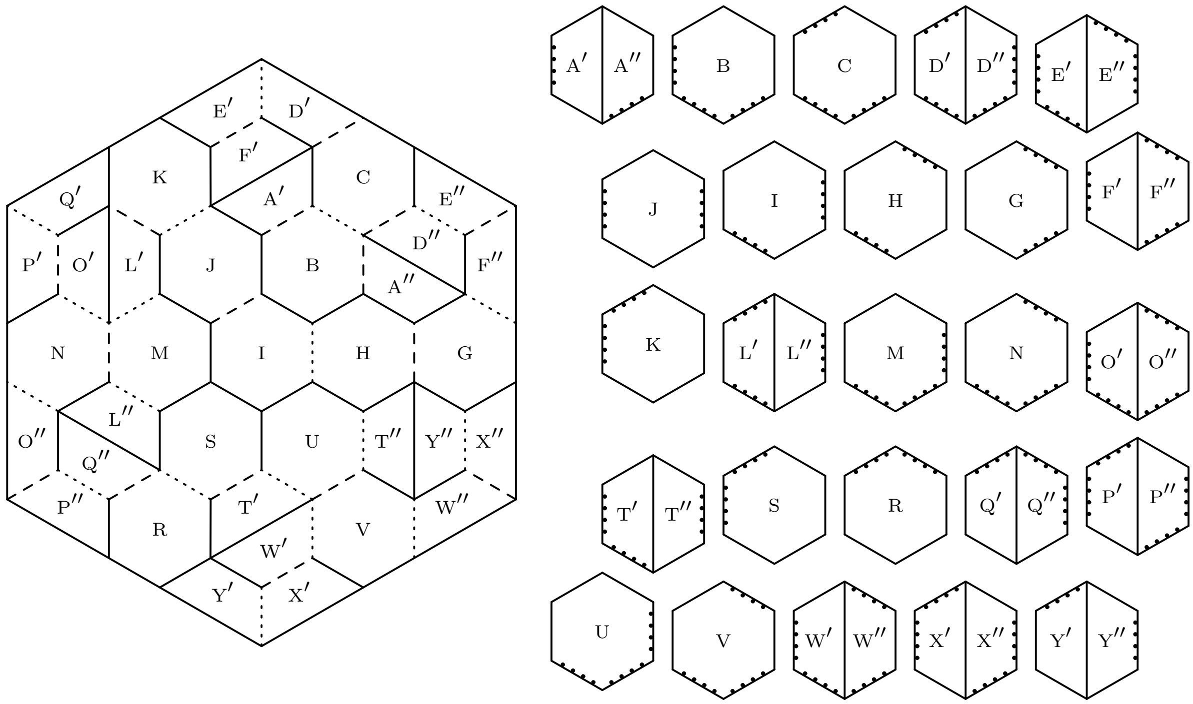

Figure 8 : *Folding dissection of a 1-level hexagon to a 25-level hexagon*

is less symmetrical than the previous dissection, but still winds around until all hexagons have been hinged. This gives the 37-piece folding dissection in Figure 8.

As the number of levels, the number of pieces, and the number of folds get large, it helps to check these schemes with either paper or wooden models. Yes, I have taped together several wooden models, using pieces laser-cut from thin cherry wood. It can be a bit of a challenge to tape pieces together correctly, and performing the folding to convert from many levels to one level takes some concentration, though the artfully rendered result is worth the effort. Is it possible to continue this series with a 51-piece folding dissection of 1-level hexagon to a 36-level hexagon, followed by a 67-piece folding dissection of 1-level hexagon to a 49-level hexagon, etc.? More than likely it can be done, if one has the patience to figure it out.

It is also possible to handle the 6-pointed star $\{6/2\}$. I previously described a 13-piece folding dissection of a 1-level $\{6/2\}$ star to a 3-level $\{6/2\}$ star in [5].

3 Folding dissections of multi-level regular dodecagons and dodecagrams

Next up are folding dissections of 12-pointed figures with lots of gorgeous symmetry. The first is a folding dissection of a 4-level dodecagon to a 2-level dodecagon. It is based on a 20-piece folding dissection of two 2-level dodecagons to one 2-level dodecagon (see [4]). Essentially, I stacked one of the small 2-level dodecagons on top of the other, and then glued two pieces together to get a 19-piece stack-folding dissection. The result, with 3-fold rotational symmetry, appears in Figure 9. The dotted edges indicate tape on the top side of the prisms, and the dashed edges indicate tape on the bottom side of the figures.

My dissection draws inspiration from Lindgren's 10-piece unhingeable dissection of two dodecagons to one in [7]. Those familiar with Lindgren's dissection will recognize the shape of pieces A, D, G, M, P, and S. They may also notice that I divide the rest of the levels of the small dodecagons that contain those pieces into pieces that are somewhat reminiscent of pieces from Lindgren's dissection. But here the fun begins:

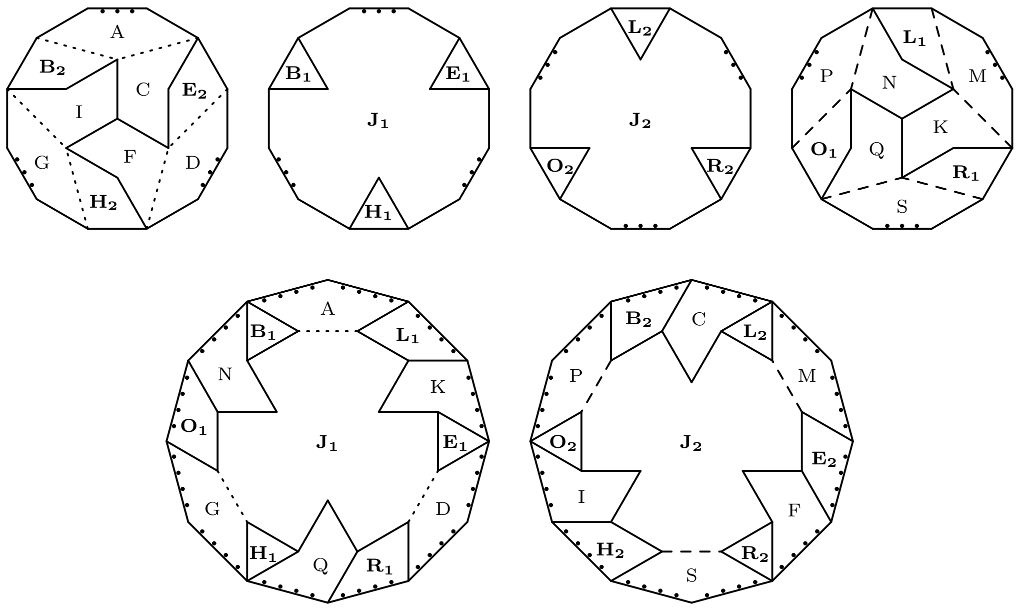

Figure 9 : *Folding a 4-level dodecagon to a 2-level dodecagon*

Cut small equilateral triangles from piece *J* in the interior of the tall dodecagon, and attach those triangles to certain pieces, creating six pieces, each of which appears on two levels and is labeled with emboldened labels. The resulting triangular towers of pieces *B*, *E*, *H*, *L*, *O*, and *R* help to fill in the rest of the outer ring of the short dodecagon.

Note that the new dissection is rounded, in the sense defined on page 14 of my book on piano-hinged

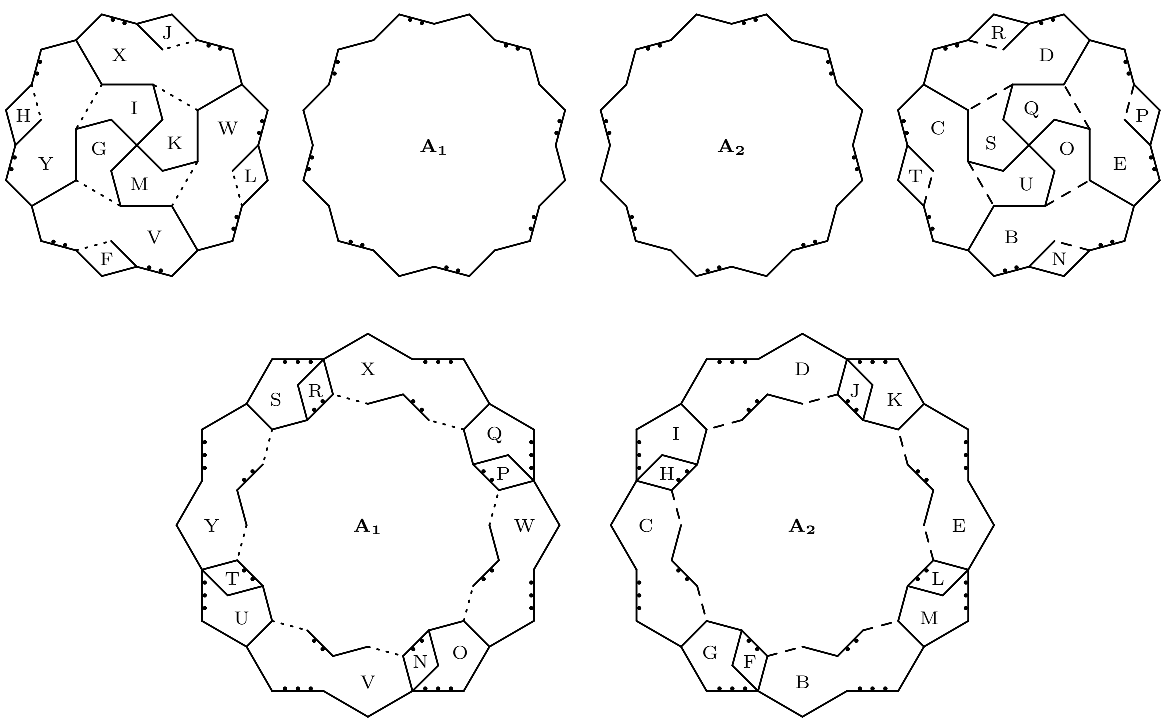

Figure 10 : *Folding dissection of a 4-level dodecagram {12/2} to a 2-level {12/2}*

dissections [3]. Rounding removes a small volume of material along an edge because the trailing edge of some rotating piece would otherwise bind against a neighboring piece. For example, the triangular tower of piece *B* in Figure 9 will bind against piece *J* as we rotate piece *B* on the hinge joint that it shares with piece *A*, unless the trailing edge of *B* is rounded. Pieces *E*, *H*, *L*, *O*, and *R* must be handled similarly.

Moving on to the dodecagram {12/2}, we find a great opportunity for a folding dissection, by making use of the 13-piece dissection of one {12/2} to two such stars in [1]. If we stack two large {12/2}s on top of each other, then we see that the four moderately large pieces on the lower level (*B*, *C*, *D*, and *E*) can fold down to form a skeleton of the lowest level of the 4-level {12/2}. This mimics what happens in the folding dissection of two squares to one in Figure 1. Note that we have labeled those pieces with the same letters. Not unsurprisingly, the top level of the 2-level {12/2} similarly will have its four moderately large pieces (*V*, *W*, *X*, and *Y*) fold up to form the outline of the top level of the 4-level {12/2}.

At first, the remaining pieces seem to be out of place, but careful cutting and folding of the remaining pieces in the 2-level {12/2} fills in the remaining spots on the top and bottom levels of the 4-level star. For example, on the bottom level of the 2-level {12/2} pieces *F* and *G* get hinged to piece *V*, and then land with piece *V* on the top level of the 4-level {12/2}. This results in the 25-piece folding dissection of Figure 10, where the middle two levels of the 4-level star comprise one piece *A* and whose labels are emboldened. Note that each of the eight pieces hinged to piece *A* is hinged along two separate line segments. Also, there is both 4-fold rotational symmetry and symmetry with respect to top and bottom.

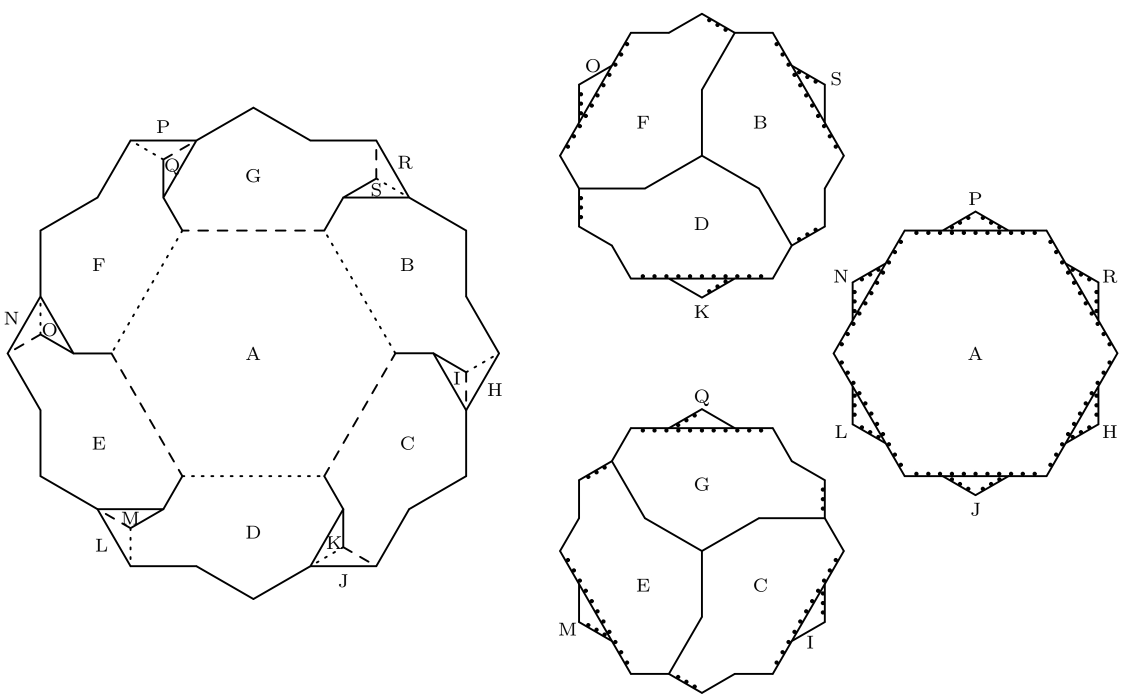

Figure 11 : *Folding dissection of a 1-level dodecagram {12/2} to a 3-level {12/2}*

Even easier, perhaps, is the 19-piece folding dissection of a 1-level {12/2} to a 3-level {12/2} in Figure 11. We split the top level of the 3-level star into six pieces, with three of them folding down to fill out some of the outline of the 1-level star. We similarly split the bottom level into six pieces, with three of them folding up to fill out more of the outline of the 1-level star. This is similar to the folding dissection of a 1-level hexagon to a 3-level hexagon, which we have seen in Figure 2. However, we need to handle appropriately the extra “bumps” on the boundary of the dodecagrams. Serendipitously, these bumps can fold so that they fill out the remaining boundary of the 1-level star. The folding does not necessitate rounded pieces. Starting with the 1-level star, fold on the the hinge between pieces *G* and *R* (which is conveniently orthogonal to the boundary between pieces *B* and *S*), then fold on the hinge between pieces *R* and *S*, and handle the other isosceles triangles similarly. This frees up pieces *B* through *G* to fold around piece *A*. Finally fold pieces *H* through *S* into their final positions in the 3-level star, which has 3-fold rotational symmetry.

4 Conclusion

We have created folding dissections for hexagons, dodecagons, and dodecagrams. The lovely symmetry and beautiful geometry make them obvious candidates for both physical models and animations. Our folding dissections of a 1-level hexagon to a 4-level hexagon, to a 9-level hexagon, to a 16-level hexagon, and to a 25-level hexagon give a glimpse of the first four of a probably infinite sequence of such dissections. We have assembled models for the 16-level hexagon and the 25-level hexagon, using 3/16-inch cherry wood that is laser-cut and then taped together using a strong, clear industrial tape from the 3M company. The resulting models are both kinetic art and elegant folding puzzles.

The folding dissections of the dodecagon and the dodecagram {12/2} are charming curiosities. I created animations of the dodecagon and dodecagram folding using the C++-based Open Geometry system, which produced a sequence of bitmaps that I converted into an mp4 file by using a Unix script (See `http://www.cs.purdue.edu/homes/gnf/book3/br2013.html`). Note the use of simple trigonometry to speed-up and slow-down the movement, producing a fluid motion. I used no software to discover or verify the designs, relying instead on artistry and my knowledge of the underlying geometry. As noted previously, the folding dissections of the {12/2}s in Figures 9 and 10 are elaborations of the simpler folding dissections of squares and of hexagons in Figures 1 and 2, respectively. Can further such examples be found?

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