Exploring Braids through Dance: The "Waves of Tory" Problem

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Abstract

Despite obvious links between many forms of dance and mathematics, the relations between them are rarely discussed. In this paper, we explore one connection between dancing and mathematics – braids. The bijection between dance and braiding enables us to kinesthetically teach various concepts in group theory. We describe an interactive workshop wherein the participants gain a stronger understanding of group theory by collaboratively dancing braids.

Introduction

Traditional set dancing has inherently geometric and mathematical underpinnings. Each dance consists of a group of people moving in predetermined patterns such that each person ends up at a designated place at a designated time. Typically, each dancer in a dance will have a designated "home" position that they return to several times throughout the dance.

Despite the obvious links, the relationship between dance and mathematics is rarely discussed. In this paper, I explore one connection between dancing and mathematics – braids. I describe an interactive workshop wherein the participants dance while holding ribbons that become braided together. Taking part in the workshop should help participants gain a stronger understanding of group theory as they collaboratively dance and "un-dance" various braids.

Motivation

My interest in dance as a braid began when a performing set dance group ran into an interesting problem with one of its choreographies – some, but not all of the people were getting back to their home positions when performing a figure known as the Waves of Tory. The reason for this was intuitive – there was a staggered start, but everybody finished the figure at the same time, so there was no way for all performers to end at their home positions. However, many of the people in my dance group still wanted a more concrete understanding of exactly what was going on in the dance [1].

In the Waves of Tory, couples form a long line and at each iteration a couple alternates between arching over the next couple or ducking under the arch of the next couple. The first couple begins by moving towards the end of the line. The remaining couples begin by moving towards the beginning of the line. Once a couple reaches the end of the line, they spend one iteration turning around to head in the opposite direction. We describe the Waves of Tory problem mathematically below.

The Waves of Tory problem

Problem: Given n couples lined up for Waves of Tory, after k bars of music at what position would the i-th couple be?

Figure 1 : *The Stanford ceili dance group performing the Waves of Tory.*

Definitions: The top couple that faces down at the start of Waves is Couple 0. The other $n-1$ couples are, from Couple 0, Couple 1, Couple 2, ... Couple $n-1$. At the start, the i-th couple is standing in the i-th position. The i-th couple starts movement on bar i.

Solution: As the dance progresses,

- for $k < i$, the couple has not yet moved and is in the i-th position
- for $k \geq 2i$, we can reduce this problem to an equivalent problem defined as follows:
  What position is the 0th couple in after m bars of music, where $m = k - 2i$?

From the 0th position facing forward, it take n bars to reach the n-th position and turn around. Similarly, from the n-th position it takes n bars to reach the front and turn around. Thus, after $2n$ bars of music, the 0th couple should return to where they began the dance. We can therefore further reduce this problem to:

What position is the 0th couple in after p bars of music, where $p = (k - 2i) \mod 2n$?[1]

- for $p < n$, the couple has been moving forward the entire time and is now in the p-th position
- for $p = n$, the couple has moved to the end, and turned around and is now in the $(n-1)$-th position
- for $p > n$, the couple has reached the end and started moving back and is now in the $(n-1)-(p-n) = (2n-p-1)$th position

After working out this solution, I realized that the movement of the dancers over time could be represented as a braid. In particular, this problem is equivalent to asking where a given strand is in a standard braid after braiding it k times. As in the Waves of Tory, a braid with n strands has each strand return "home" after $2n$ iterations. It takes n iterations for the first strand to become the last strand, and another n for it to get home. A couple that is arching over another couple is then isomorphic to a strand of the braid going in front of another, and a couple ducking underneath an arch is isomorphic to a strand going behind another.

[1] p is by definition $< 2n$

Methodology and Previous Work

The representation of dance as a braid can be done visually and dynamically as the dance progresses by having each dancer hold on to a string or ribbon. All of the strings should be attached to the same place – preferably around ceiling level, such that the dancers do not become tangled in the string while dancing. In this way, the movement of the dancers around each other will form braids that are representative of their relative movements over time.

The concept of modeling the movement of dancers in this way is not new. The traditional maypole dance takes explicit advantage of the fact that the movement of a group of people in a dance can be modeled through over time as a braid. Dancers connected to ribbons move around a pole such that they first form a traditional n-strand round braid and then reverse their patterns to unbraid it. Richeson describes the variety of braids that could be formed by dancing around a maypole as forming the "maypole braid group" [6]. This type of braid is generally known as an annular braid in the literature [4]. In contrast, the dance representation that I describe does not involve dancing around a pole, and, thus, the braids generated are not necessarily annular braids as in the maypole dance.

John Conway has also described a relationship between knots and dance in his work on tangles [2, 3]. In particular, he describes a square dance wherein one pair of dancers hold the ends of one rope and the other pair hold the ends of the other rope. The movement of the dancers then causes the rope to tangle. This differs from the exercise that I describe in that each rope is held by not one, but two dancers and the movement does not as easily represent the movement of the dancers over time.

Ward-Penny gives an overview of various cross-curricular teaching methods for teaching mathematics through dance and movement, including, most relevantly, the work of Watson, who identifies spacial exploration, rhythm, structure, and symbolisation as characteristics in common between dance and mathematics [7, 8]. Watson suggests teaching about inverse functions simply by having the students be required to all return to the same home position after a fixed number of counts, rather than by requiring that the braid be completely untangled, as is suggested by the representation of a dance as a braid.

Workshop Description

In our exercise, we will explore dancing different braid patterns. This bijection between dance and braiding enables us kinesthetically teach various concepts in braid and group theory.

Participants begin the workshop by experimenting with very simple operations. At any point in time during a dance, two people can swap places, or perform a swap operation. We will define a swap where the dancer on the left passes in front of the dancer on the right to be a positive swap, and a swap where the dancer on the left passes behind the dancer to his right to be a negative swap. Observe that a single positive swap can be composed with a single negative swap between the same two dancers to end up in the original configuration. Thus, a negative swap is the inverse operation of a positive swap.[2]

Participants continue by experimenting with the braids that they can form by holding onto the ribbons and performing various operations. They should be able to quickly understand how to make a simple three strand braid by weaving around each other, as in Figure 2. They should then be encouraged to form similar braids with more and more people. Participants may find it easier to dance to music, performing 0 or 1 swap operations per standard count (depending on the speed of the music and distance of the dancers, this may be per beat or per measure).

A few interesting braid patterns are the n-strand classic braid, the n-strand trick braid[3], and mixed

[2] In standard notation, this is the trivial relation $\sigma_i \sigma_i^{-1} = e$

[3] This braid is known as a trick braid as, for odd n, one can form it with both the tops and bottoms of all the strands tied together.

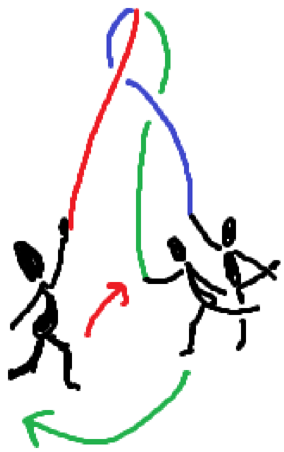

Figure 2 : *Dancers forming a three strand braid.*

braids. Examples of these patterns can be seen in Figure 3.

The n-strand classic braid is the braid that most people know. Dancers can form this braid by alternating going in front and behind of the next dancer on each count until they reach an end of the line. At this point they should turn around for one count and repeat the pattern in the other direction.

The n-strand trick braid is similar, but instead of alternating in front and behind movement, the dancers go in front of all the other dancers until they reach the middle, at which point they go behind all of the other dancers until they reach the end.

A mixed braid goes back and forth regularly between multiple different braid patterns. These patterns need not be on the same number of strands. For example, the 3 in 5 strand mixed braid depicted in Figure 3 alternates between a 3-strand classic braid and a 5-strand classic braid every two counts. To duplicate this, have the dancers alternate going in front and behind as in a normal classic braid. However, upon reaching the end of a line, instead of immediately turning around for one count, they should instead pause for two counts before turning around. Dancers may take this time to spin around or perform a "solo". When a dancer reaches a position adjacent to an end position, they should continue as in a regular 5-strand braid iff the dancer on the end is ready to swap with them. Otherwise, they should spend one count turning around and continue as usual in the other direction. The class of mixed braids is infinite, and it is an interesting exercise to come up with simple rules to generate them in a dance.

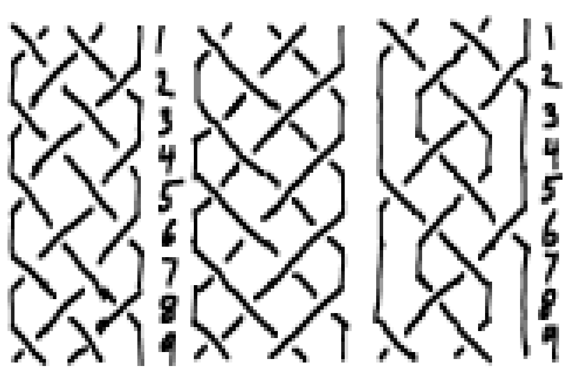

Figure 3 : *Nine counts of a 5-strand classic braid, a 5-strand trick braid, and a 3 in 5 mixed braid.*

It will be interesting to see if the participants can figure out how long it takes for each of these patterns to return all of the dancers to their original positions. In addition, the dancers should be encouraged to experiment with patterns not described above. Is there a pattern to how they should weave to make braids that "look nice"? Are these patterns correlated to how attractive they find the dance itself to be?

As the participants try these exercises they should find themselves naturally searching for and finding the inverse functions for their braid dances. For a workshop with younger children, the entire workshop may focus on gaining understanding of these basic patterns and inverses.

In a more advanced workshop, we can more explicitly explore basic braid and group theory. Basic group theory concepts such as generators, relations, cyclic groups, and abelian groups can be explained using braid danced examples, as described below.

We will use the standard terminology of σ_i to represent a positive swap between the i-th and the $(i+1)$-th dancer and σ_i^{-1} to represent a negative swap. As these are the only possible operations, it is obvious that any braid group B_n is generated by the operations $\sigma_1 \ldots \sigma_{n-1}$ and the inverse operations $\sigma_1^{-1} \ldots \sigma_{n-1}^{-1}$. Similarly, we will use the standard terminology B_n to represent the braid group on n strands [5].

We can see that the braid group formed by one dancer is trivial – a single dancer cannot form any braids by themselves. Similarly, the braid group by two dancers is an infinite cyclic group and is thus isomorphic to the integers under addition. Participants can also use dancing to help intuit properties for arbitrary braid groups, B_n. For example, participants can compare the results of $\sigma_1\sigma_2$ followed by $\sigma_2\sigma_1$ to see that B_n for all $n > 2$ is non-abelian.

In contrast, it is clear that $\sigma_i\sigma_j$ is equivalent to $\sigma_j\sigma_i$ where $|i - j| > 1$. The relation between $\sigma_n\sigma_{n+1}\sigma_n$ to $\sigma_{n+1}\sigma_n\sigma_{n+1}$ is less obvious, but can be seen clearly when drawn, as in Figure 4, or danced. The participants should experiment with dancing these relations.

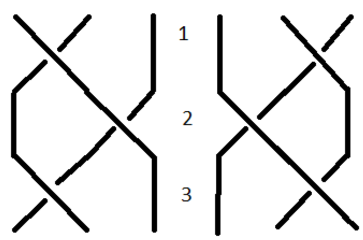

Figure 4: $\sigma_1\sigma_2\sigma_1$ *(left) is equivalent to* $\sigma_2\sigma_1\sigma_2$ *(right).*

After completing these exercises, the instructor should share with the participants that all braid groups can be defined using the generators mentioned previously and just these relations. The generalized presentation for a braid group on n strands is:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_j\sigma_k = \sigma_k\sigma_j \rangle, \text{ where } i \leq n-2 \text{ and } |j-k| > 1$$

Participants should find that they have a better intuitive understanding of this presentation after this workshop.

Conclusion

In this paper we extend naturally from traditional set dance to the connection between dance, braids, and mathematics. Our bijection between dance and braiding enables us to teach various concepts in group theory through choreographed movement. We describe a workshop that explicitly visualizes this connection by having each participant dance while holding onto a ribbon. Of particular note are the concept of performing an inverse dance to end up back at the original state and the connections between these dances and group theory.

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