Analytical Calculation of Geodesic Lengths and Angle Measures on Sphere Tiling of Platonic and Archimedean Solids

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Abstract

There are five Platonic solids and thirteen Archimedean solids, and they have many interesting characteristics. One of them is that their faces can be projected outward to a circumscribing sphere, producing tilings of the sphere. In this paper we show how to use analytical methods to calculate the lengths of the geodesics and the measures of the angles for these tilings.

Introduction

In art such as modular origami and architecture, regular and semi-regular polyhedra have been popular subjects [2][3]. These polyhedra have regular polygons as their faces and edges with the same length. Five Platonic solids and thirteen Archimedean solids in Figure 1 are convex regular and semi-regular polyhedra [1]. One of interesting properties of these solids is that all the vertices are on the sphere that circumscribes the solid. The shortest distance on the sphere's surface between any two adjacent points is obtained by the arc of a great circle. By the radial projection of edges of a polyhedron onto the surface, we get arcs which are called geodesics. These geodesics define a uniform tiling for each solid as in Figure 2. In this paper we analytically compute the length of a geodesic and interior angle measures of spherical polygons on the tiling of all the Platonic solids and Archimedean solids.

Necessary Tools for Calculation

Spherical polygon and spherical polyhedron. A bounded region partitioned by arcs on a great circle is called a spherical polygon. The radical projections of edges are great arcs and a spherical polygon is acquired for each face of a solid. A spherical polyhedron is a tiling of a sphere where the surface is divided into spherical polygons. Just like a planar polygon, a spherical polygon has interior angles, and an angle on a sphere can be defined as an angle between two tangent lines of arcs. Also, the length of a side is specified by the angle at the sphere's center subtended to the endpoints of the sides [4].

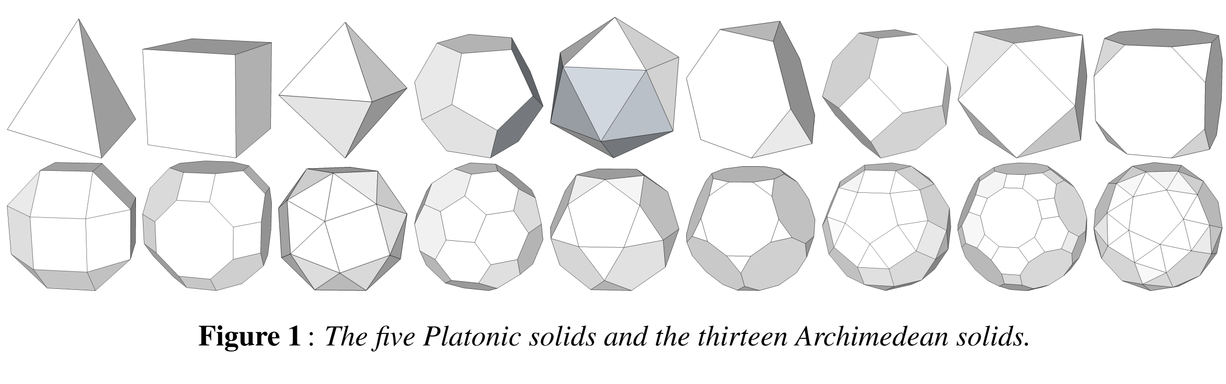


Figure 1 : *The five Platonic solids and the thirteen Archimedean solids.*

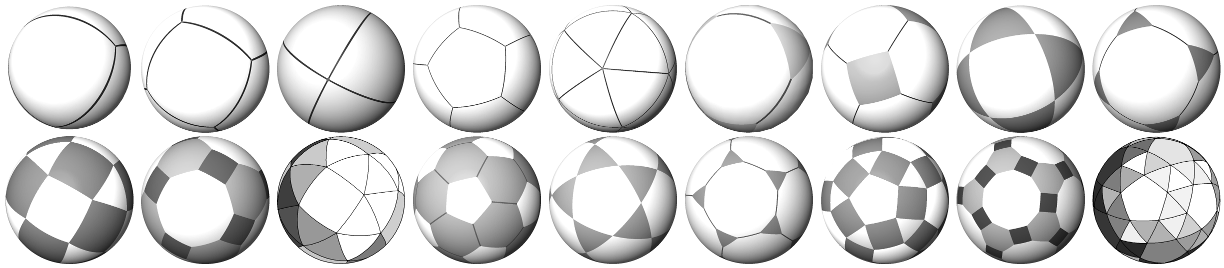


Figure 2 : *Tilings of the sphere arising from the Platonic and Archimedean solids.*

Spherical trigonometry. There are spherical trigonometry identities that are similar to ones in planar geometry. Let a, b, and c be three sides of the triangle and α, β, and γ be three angles as in Figure 3(*a*).

Law of cosine rules for sides $cos(a) = cos(b)cos(c) + sin(b)sin(c)cos(\alpha)$

Law of cosine rules for angles $cos(\alpha) = cos(\beta)cos(\gamma) + sin(\beta)sin(\gamma)cos(a)$

The law of sines, tangents, half angle formulas, and other rules are well-known. See, e.g., [4].

Napier's rules for right spherical triangle. Napier's rules for right spherical triangle can be easily used with a circle notation. The circle is divided into five sectors, and all the angles of the triangle are labeled in their circular order except for the right angle. For an angle that is not adjacent to the right angle, its complement angle is used. Once the circle is built as in Figure 3(*b*), for any choice of three angles, the sine of the middle angle is equal to the product of the cosines of the opposite angles or the product of the tangents of adjacent angles. A right spherical triangle can be formed for any right polygon in a tiling by using the center of the polygon, a midpoint of an edge, and a vertex [6].

Cubic formula. The last mathematical tool is the general formula for the roots for a cubic equation. In a couple of cases (snub cube and snub dodecahedron), the length of a geodesic is a real root of a cubic function.

Calculation

Assuming the radius of the sphere is 1, the circumference of a great circle is 2π. Since there is only one type of spherical polygon for a given Platonic-solid tiling, it is easy to calculate lengths and angles. For each Archimedean solid, a careful investigation was carried out to build a solvable system. In the following sections, x is the length of a geodesic, and α, β, and γ are angles of spherical polygons.

Tetrahedron and icosahedron. A spherical triangle created by a tetrahedron is shown in Figure 3(*c*). The angle measure is $2\pi/3$ since three faces meet at a vertex. Then we can use the law of cosines for sides as in the following.

$$cos(x) = cos(x)cos(x) + sin(x)sin(x)cos(2\pi/3)$$

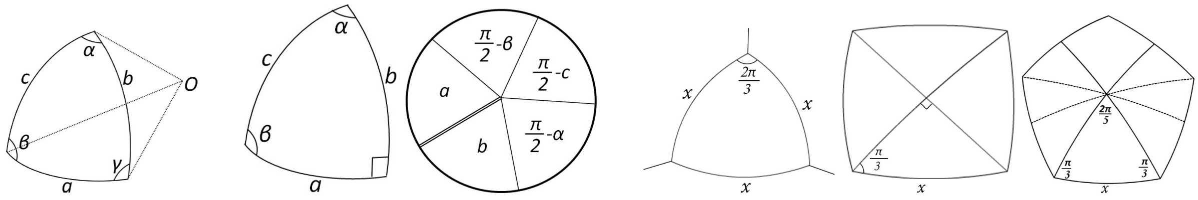
By solving this for $cos(x)$ using trigonometric identities, we can get $cos(x) = -1/3$ or 1. Then $x = cos^{-1}(-1/3) = 1.9106$. For an icosahedron, the same method can be used with an angle measure $2\pi/5$. The length of the geodesic is 1.1072

Cube and dodecahedron. Figure 3(*d*) and (*e*) show a tiled square and pentagon. The law of cosines for angles can be used for a triangle. For a cube,

$$cos(\pi/2) = cos(\pi/3)cos(\pi/3) + sin(\pi/3)sin(\pi/3), \qquad x = cos^{-1}(1/3) = 1.2310$$

For a dodecahedron

$$cos(2\pi/5) = cos(\pi/3)cos(\pi/3) + sin(\pi/3)sin(\pi/3), \qquad x = cos^{-1}(0.7754) = 0.7297$$



(a) (b) (c) (d) (e)

Figure 3 : *Spherical triangle (a), Napier's circle (b), and spherical polygons on tiling (c, d, e).*

Octahedron. Since four geodesics makes a great circle, $x = {}^{\pi}/_{2}$. $\alpha = {}^{\pi}/_{2}$ since four faces meet at a vertex.

Truncated tetrahedron and seven other solids. The vertex configuration of a truncated tetrahedron is $(3, 6, 6)$ [1]. If we let α be the interior angle of a hexagon and β be the interior angle of a triangle, then

$$2\alpha + \beta = 2\pi. \tag{1}$$

Also from right triangles in a hexagon and a triangle, Napier's circles can be built as in Figure 4(a) and (b). By selecting three angles that include α, β, and x, we can create two more equations.

$$sin({}^{\pi}/_{3}) = cos({}^{\pi}/_{2} - {}^{\alpha}/_{2})cos({}^{x}/_{2}), \qquad sin({}^{\pi}/_{6}) = cos({}^{\pi}/_{2} - {}^{\beta}/_{2})cos({}^{x}/_{2}). \tag{2}$$

We can find α, β, and x by solving the system of equations from (1) and (2). The solution is $x = 0.8812$, $\alpha = 2.5559$, and $\beta = 1.1714$

Because seven Archimedean solids have two types of regular polygons, similarly we can find α, β, and x. They are categorized according to the first equation between interior angles.

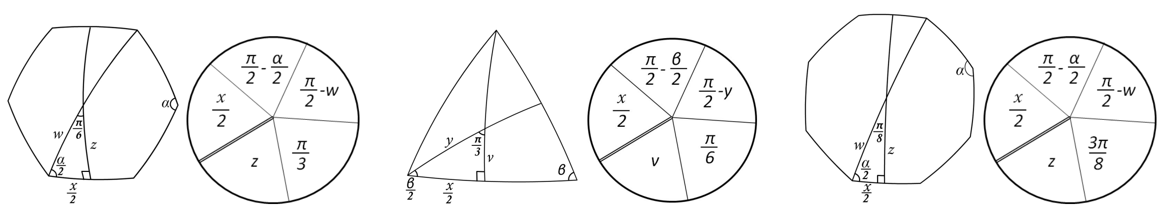
- $2\alpha + \beta = 2\pi$: truncated cube (3, 8, 8), truncated octahedron (4, 6, 6), truncated dodecahedron (3, 10, 10), truncated icosahedrons (5, 6, 6)
- $2\alpha + 2\beta = 2\pi$: cuboctahedron (3, 4, 3, 4), icosidodecahedron (3, 5, 3, 5)
- $3\alpha + \beta = 2\pi$: rhombicuboctahedron (3, 4, 4, 4)

Great rhombicuboctahedron and two other solids. A great rhombicuboctahedron has three different polygons as its faces; squares, hexagons, and octagons. Its vertex configuration is (4, 6, 8). So we introduce three unknowns, α, β, and γ to represent interior angle measure of an octagon, a hexagon, and a square respectively. According to the vertex configuration, we have $\alpha + \beta + \gamma = 2\pi$. As in Figure 4($c$), Napier's circle can be constructed. Three angles with α, x will derive a similar equation as (2).

$$sin({}^{3\pi}/_{8}) = cos({}^{\pi}/_{2} - {}^{\alpha}/_{2})cos({}^{x}/_{2})$$

From Napier's circle for a hexagon and a square,

$$sin({}^{\pi}/_{3}) = cos({}^{\pi}/_{2} - {}^{\beta}/_{2})cos({}^{x}/_{2}), \qquad sin({}^{\pi}/_{4}) = cos({}^{\pi}/_{2} - {}^{\gamma}/_{2})cos({}^{x}/_{2})$$



(a) (b) (c)

Figure 4 : *Right triangles on three spherical polygons and corresponding Napier's circles.*

By solving the system of equations, we can find $x = 0.4349$, $\alpha = 2.4823$, $\beta = 2.1812$, and $\gamma = 1.6197$. For two other solids with three types of faces, the same method can be used to find solutions.

- $\alpha + \beta + \gamma = 2\pi$: great rhombicosidodecahedron (4, 6, 10)
- $\alpha + 2\beta + \gamma = 2\pi$: rhombicosidodecahedron (3, 4, 5, 4)

Snub cube and snub dodecahedron. A snub cube has vertex configuration (3, 3, 3, 3, 4), and the first equation is $\alpha + 4\beta = 2\pi$ where α is an interior angle of a square, β is an interior angle of a triangle. From right triangles we have

$$sin(\pi/4) = cos(\pi/2 - \alpha/2)cos(x/2), \qquad sin(\pi/6) = cos(\pi/2 - \beta/2)cos(x/2)$$

By eliminating x, we can get $sin(\alpha/2) = \sqrt{2}sin(\beta/2)$.

After substituting $\alpha/2$ with $\pi - 2\beta$, and simplifying using trigonometric identities, we have a cubic equation in terms of $cos(\beta/2)$, $8cos^3(\beta/2) - 4cos(\beta/2) - \sqrt{2} = 0$. Using the cubic formula, we have $cos(\beta/2) = 0.8425$. The solutions are $x = 0.7628$, $\alpha = 1.7320$, and $\beta = 1.1378$. A snub dodecahedron has a similar vertex configuration, (3, 3, 3, 3, 5), and the solution can be found the exactly same way.

Summary and Conclusion

In this paper, for all eighteen Platonic and Archimedean solids which are a popular subject in art and architecture, the length of geodesics and interior angle measures on their spherical tilings were calculated using analytical method. The results are summarized in the following table. When there is more than one polygon type, the interior angle measure of the polygon with more edges is shown first. Numerous data about Platonic solids and Archimedean solids can be found in some literature such as [5]. Using central angles for an edge and (edge length)/(circumradius), it is possible to verify these calculations.

Solid	Vertex Configuration	Length of Geodesic (x)	Angle1 α	Angle2 β	Angle3 γ	Solid	Vertex Configuration	Length of Geodesic (x)	Angle1 α	Angle2 β	Angle3 γ
Tetrahedron	3, 3, 3	1.9106	2.0944			Rhombicuboctahedron	3, 4, 4, 4	0.7310	1.7176	1.1304	
Cube	4, 4, 4	1.2310	2.0944			Great Rhombicuboctahedron	4, 6, 8	0.4349	2.4823	2.1812	1.6197
Octahedron	3, 3, 3, 3	1.5708	1.5708			Snub Cube	3, 3, 3, 3, 4	0.7628	1.7320	1.1378	
Dodecahedron	5, 5, 5	0.7297	2.0944			Truncated Icosahedron	5, 6, 6	0.4064	2.1696	1.9440	
Icosahedron	3, 3, 3, 3, 3	1.1072	1.2566			Icosidodecahedron	3, 5, 3, 5	0.6288	2.0344	1.1072	
Truncated Tetrahedron	3, 6, 6	0.8812	2.5559	1.1714		Truncated Dodecahedron	3, 10, 10	0.3386	2.6096	1.0640	
Truncated Octahedron	4, 6, 6	0.6436	2.3002	1.6828		Rhombicosidodecahedron	3, 4, 5, 4	0.4460	1.9571	1.6228	1.0805
Cuboctahedron	3, 4, 3, 4	1.0472	1.8925	1.2490		Great Rhombicosidodecahedron	4, 6, 1	0.2633	2.5697	2.1248	1.5884
Truncated Cube	3, 8, 8	0.5704	2.5936	1.0961		Snub Dodecahedron	3, 3, 3, 3, 5	0.4680	1.9643	1.0797	

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