Using Star Polygons to Understand Cyclic Group Structure

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Abstract

The study of abstract groups is a challenging one for many undergraduate students. In this paper, we present an overview of a geometric context which can help students interpret and understand the structure of cyclic groups, as well as gain a broader conceptual understanding of such central ideas as subgroups and cosets. The authentic artistic context of star polygons can both motivate students to persist in a search for patterns and provide a concrete referent from which students can generalize algebraic ideas.

Introduction and Context

Although abstract algebra (the study of abstract groups, rings, and fields) is a common required course for undergraduate mathematics majors, evidence exists that student learning of key algebra concepts from these courses is less than satisfactory [1, 2]. Furthermore, the Conference Board of the Mathematical Sciences has recommended that all future secondary teachers be required to complete at least one course in abstract algebra [3] in which these future teachers develop a conceptual grasp of group structure, but notes that currently, "Unfortunately, too many prospective high school teachers fail to understand connections between [abstract algebra and number theory] and the topics of school algebra" (p. 40). Compounding the problem, little pedagogical research has been conducted to investigate the sources of or potential solutions to students' difficulty in mastering abstract algebra ideas [4]. In this paper, I will present one activity that can be used to help students understand key concepts of abstract group theory, such as subgroup, generator, and coset. Because it has been suggested that the ideas of group, subgroup, and coset are among the most fundamental ideas of abstract algebra [2], an activity that can help students understand and master these ideas could be useful for many different course settings.

In particular, this activity was used with an abstract algebra class for practicing secondary teachers enrolled in a Master's program in Mathematics Education focusing on symmetry groups. Students in this course had varying mathematical backgrounds and abilities, and they insisted on gaining a conceptual understanding of key ideas before moving on. Prior to this lesson, students had studied examples of finite groups, focusing especially on groups of integers ($\boldsymbol{Z}_n$) and isometry groups. Students were familiar with the properties of dihedral groups and could find subgroups of dihedral groups and generators of those subgroups, but still lacked confidence and conceptual clarity in the strategies they used to do so. Just prior to this activity, the idea of a cyclic group C_n was introduced by considering of the orientation-preserving elements of the dihedral group D_n (that is, the rotational symmetries of a regular n-gon).

The purpose of this activity is to deepen students' understanding of cyclic groups C_n, provide them with an intuitive understanding of the conceptual undpinnings of strategies for finding generators and subgroups of C_n, and to introduce the idea of a coset through a geometric representation. In order to achieve these purposes, we use the idea of *star polygons*. Star polygons are a familiar motif in art, especially Islamic art, and n-pointed stars have been of interest to mathematicians and artists since ancient times. For example, the 5-pointed star was of special significance to the ancient Greeks for its mysterious properties and relationship to the golden ratio and also appears in ancient art of the Celts, Hebrews, Druids, Egyptians, and others [5]. Using this authentic artistic context provides a motivation for students to persist in looking for patterns and also offers an excellent geometric representation of these important group theory ideas.

Steps of the Activity

To introduce the activity, students are given a set of circles with different numbers of equally-spaced points (ranging from four points to twelve) and are asked to connect every k^{th} point for all choices of k. For instance, a traditional 5-pointed star can be formed by connecting every 2^{nd} point of five equally spaced points around a circle, so it is referred to as a (5, 2) star. Students work in groups to connect points and form stars for each set of n points and then look for patterns. The patterns that students often notice include the fact that an (n, k) arrangement and an $(n, n - k)$ arrangement are identical. Students also tend to notice that all points on the circle will only be connected when n and k are relatively prime (as in Figure 1 below). In group theory terms, the first point represents the group generator a and each k^{th} repetition of the point represents its composition a^k.

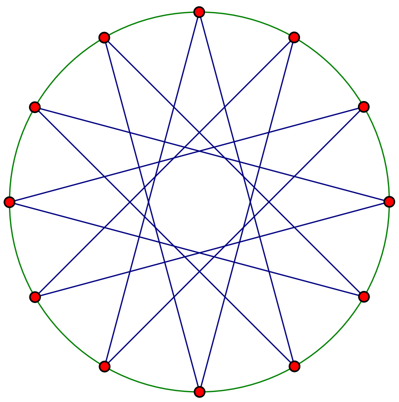

Figure 1. *Star polygon (12,5) illustrates that a^5 is a generator of C_{12}*

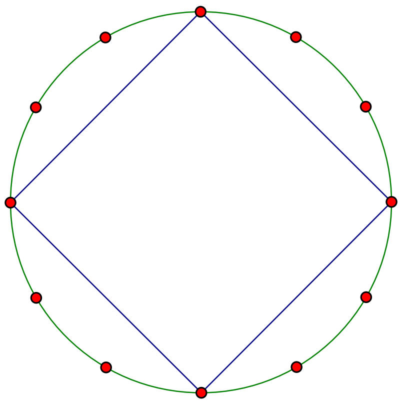

Figure 2. *"Star" polygon (12,3) illustrates the subgroup of C_{12} that is isomorphic to C_4*

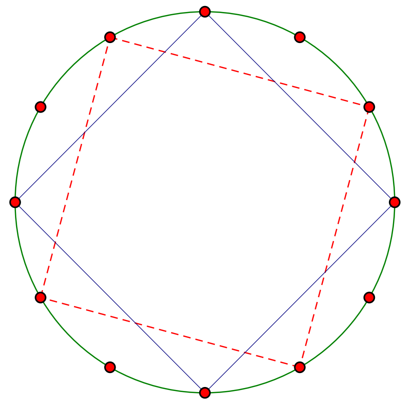

Figure 3. *"Star" polygon (12,3) and representation of a coset formed by composing with a^2*

This much of the activity could be presented to secondary students as easily as secondary teachers, so additional questions must be posed to help students connect the geometric patterns they see to group structure. For example, by asking students to consider what choices of n and k will allow all points to be included in the star and how that relates to group structure, instructors can prompt students to make a connection between relative primeness and group generators of cyclic groups (see Figure 1 above for an example). For choices of k that are not relatively prime to n, only a subset of points will be connected. Because these points are based on a repeated composition of choosing the k^{th} point (corresponding to group element a^k), they will form a subgroup of C_n (as in Figure 2 above), leading students to the generalization that the number of subgroups of C_n is related to the number of factors of n. Finally, by wondering what might happen when each of the elements in a subgroup is composed with a different rotation, students can be introduced to the idea of a coset in a geometric way (for example, see Figure 3).

References

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