Introducing the use of the Primary Isosceles Triangles of Regular Polygons to Produce Self Similar Patterns

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Abstract

The patterns produced in this work use the basic geometry of the circle, regular polygon and similar triangles as studied in most schools. In an attempt to focus on the artistic application the mathematics will be stated and not explained in any great detail. This introduction describes the ideas and general approach. I aim to explore the symmetries of groups of polygons from an equilateral triangle up to the regular 36 sided polygon. The remainder of the paper illustrates some of the progress made to date with examples of the artwork. I have chosen a theme of early English Celtic Art as a style of decoration.

1. Introduction

Using only the basic properties of the angles in a circle and considerations of symmetry it is easy to prove the maths used in this paper, see figures 1 and 2. A regular polygon can always be dissected into isosceles triangles. What is interesting, from a self similarity point of view, is that, for each isosceles triangle, it appears that a larger similar triangle can be produced using only the original triangles, see figure 3. As a consequence any design or shape created using the original isosceles triangles can be enlarged. In addition, the resulting enlargement can be enlarged again using the same process. This procedure can be repeated any number of times. More detailed discussion can be found at [1], [2], [3], [4],[5]. Part of my interest is to investigate the ways in which the number of sides influences the shapes and symmetries of the resulting patterns. For instance, when the number of sides *n* is odd, even, prime, factors of 2, 3, 4, 5 and so on. The illustrations in this paper are mainly based on the 11-gon, 12-gon and 13-gon, 11 and 13 being prime numbers with little inherent symmetry, whereas, 12 has factors 2, 3, 4, and 6 and has more inherent symmetry, see figure 10. I have called $\boldsymbol{\theta}$ the primary angle of the n-gon. It is easy to show that the angles of all the isosceles triangles are multiples of $\boldsymbol{\theta}$ where $\theta = 180 \div n$

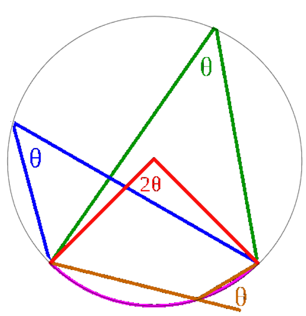


Figure 1: *central angle theorem*

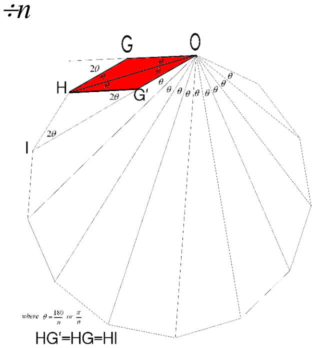


Figure 2: *The diagonals of a regular polygon*

Each of the triangles in figure 2 can be divided into isosceles triangles. This is proved by folding the polygon along a diagonal for example OH: The bit not covered by G'HO is isosceles since it is formed by two edges of the polygon, which, by definition, are equal. A primary triangle for a polygon is one whose angles are multiples of $\boldsymbol{\Theta}$. I have found it useful to designate a primary triangle as a letter followed by three integers in square brackets. For example F[2,2,7]. The numbers 2 and 7 are measures of the angles which I call frangles. The triangle F[2,2,7] is a primary triangle for the 11-gon, since 2+2+7=11 and it is an isosceles triangle since two of the frangles are the same. The use of frangles simplifies the creation of the large similar triangles, especially as n increases in size. It also simplifies the calculation of the positions of the triangles by allowing "logo" type commands, see sections 2 and 3.

2. The Primary Triangles of a Polygon

While I have an algorithm for writing down the primary triangles it is not a proof that the list is comprehensive. The list for a 11-gon is

A[1,1,9] B[1,2,8] C[1,3,7] D[1,4,6] E[1,5,5] F[2,2,7] G[2,3,6] H[2,4,5] I[3,3,5] J[3,4,4]

Figure 3 shows the dissection of the 11-gon into its 5 primary isosceles triangles, each of the frangles being a multiple of $\boldsymbol{\Theta}$. It also illustrates the precious relationship among the five triangles. Each large triangle is similar to its small equivalent with a constant enlargement factor. So far I have similar algorithms for the polygons from the 3-gon up to the 17-gon. I hope to produce algorithms up to the 36-gon if this is, in fact, possible. This should be a sufficient number to investigate some of the properties of groups of polygons. For example cases where the number of sides n is odd, even, factors of 2, 3, 4, 5, 6,a prime number, a square number and so on. Creating the large similar triangle is a manual exercise similar to completing a jig saw.

3. A Brief Description of the Software

Most modern computer languages can cope with recursive problems based on the ideas of Archimedes, Newton and many others. The software uses recursion to generate the designs. It was designed to develop pictures up to 10 layers each of up to 120 megapixels and 256 colours, which is the limit of the available hardware. this allowed for the production of good quality A0 size pictures. Large pictures could take several hours to complete so low resolution versions were initially used prior to the final version. Restart facilities were also built in. I encountered no problems with overflow errors as long as

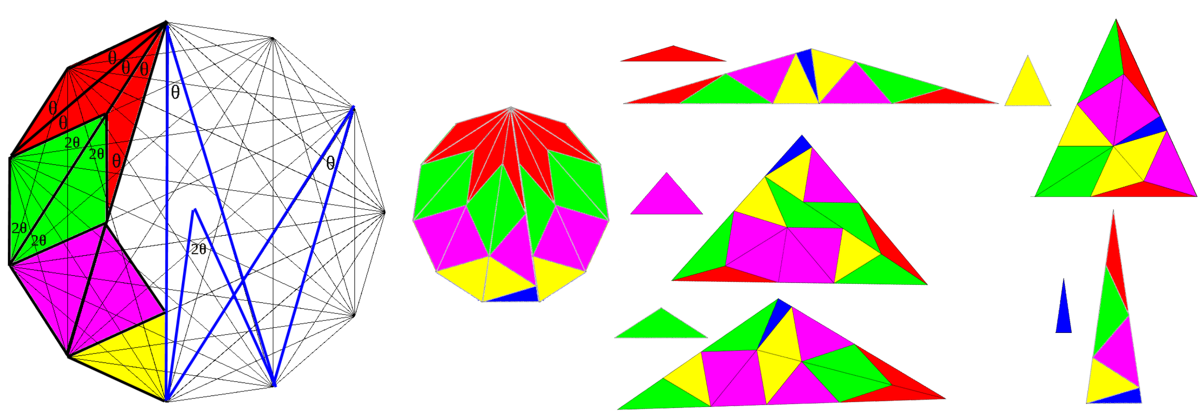


Figure 3: *The dissection of a regular polygon into its primary isosceles triangle. The enlarged version for each of the triangles are also shown.*

I kept to triangles with edge size greater than 10 pixels. In the context of 120 megapixel pictures, this was no problem. Optionally, I could include pictures into the design. This made provision for additional decoration to supplement a purely geometric design. The trigonometry associated with the formation of precious shapes was simplified by a development of turtle graphics which I used to locate the position of nodes. Instead of calculating the position of each node a virtual turtle was moved around the shape using commands like *"forward d" "right 3"* and so on - a record being kept of the turtle's position after each movement. Furthermore, by swapping the functions of left and right, both left and right handed versions were readily produced which had implications for the symmetry of the designs. Most of the pictures included were sketches of my own creation based on designs found on English Celtic bronze and ironwork.[8], [9], [10], [11]. These date from 600BCE to 200 CE.

4. Properties displayed by all Regular Polygons

The First set of pictures is typical of all regular polygons. These are illustrated using the 11-gon. The following comments refer to figure 3. The starting point for this work is the dissection of the polygons into their primary isosceles triangles. Consequently one shape that can be produced is the original polygon, At the other end a single isosceles triangle, for example triangle *I[3,3,5]*, could be the starting point. Figure 4 shows the third generation of precious transformation of the *I[3,3,5]* triangle. The outline of previous generations are shown in thick black and red. The thick black outlines the first generation of enlargement.
There is always a triangle for each polygon that has the frangles *1, 1, n-2* which, in the case of 11-gon ,is *A[1,1,9]*. Two of these can be formed into a rhombus with frangles *2, 2, 9* and *9.* (Figure 6) These rhombi can fit together around a point to form a star with *n* points. Each point can have rotational or line symmetry, depending on whether the triangles are specified as one left and one right handed triangle, or both the same handedness. Figure 6 show the two sets of triangles. They have been decorated with my sketches of typical early Celtic Art triscele motifs.
It is always possible to find a set of triangles that will fit around a point. Some of these will be cyclic, that is, each of the vertices lie on a circle, others not. To close around a point the sum of the frangles needs to total *2n*. In Figure 7 we see a design made up from the primary isosceles triangles of a 13-gon. It is decorated by a motif from an early Celtic bucket.

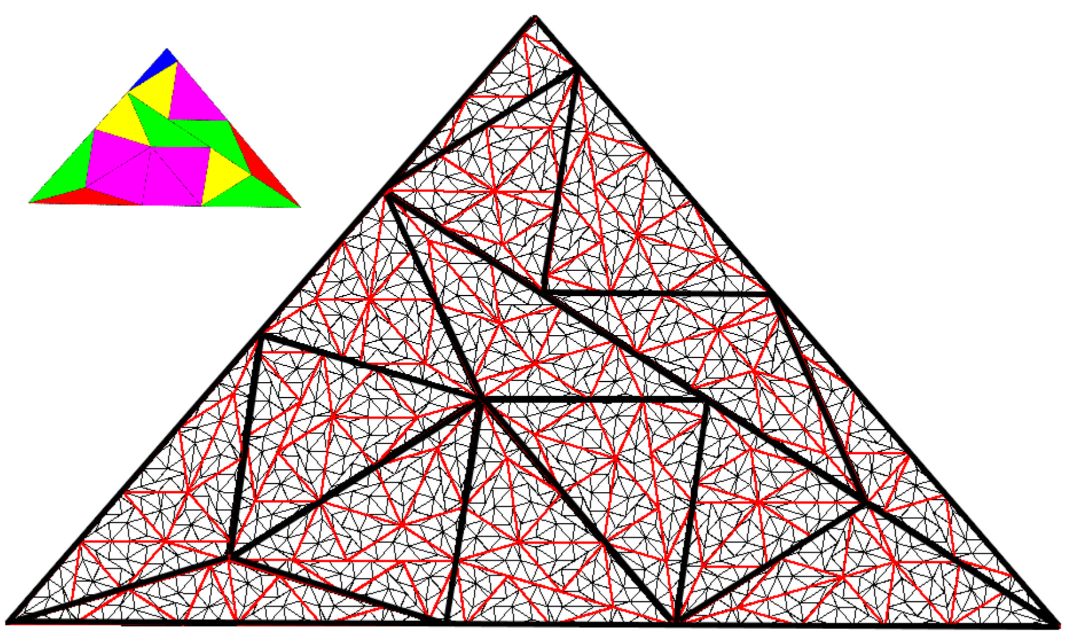


Figure 4: *Illustration of three generations of an I[3,3,5] triangle.*

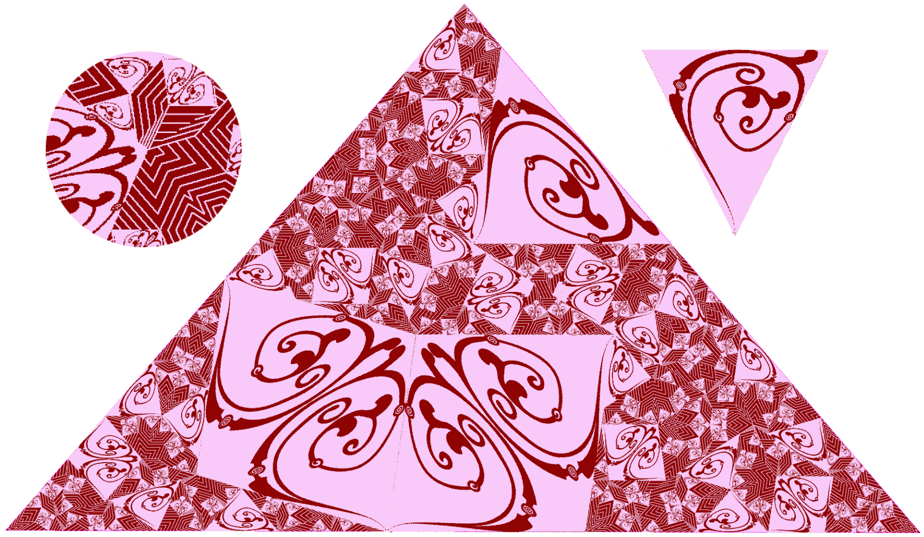


Figure 5: *This single triangle based on the 11-gon has been decorated using a typical Celtic motif in all the I[3,2,5] triangles. The remaining triangles are filled with hatching., See inset*

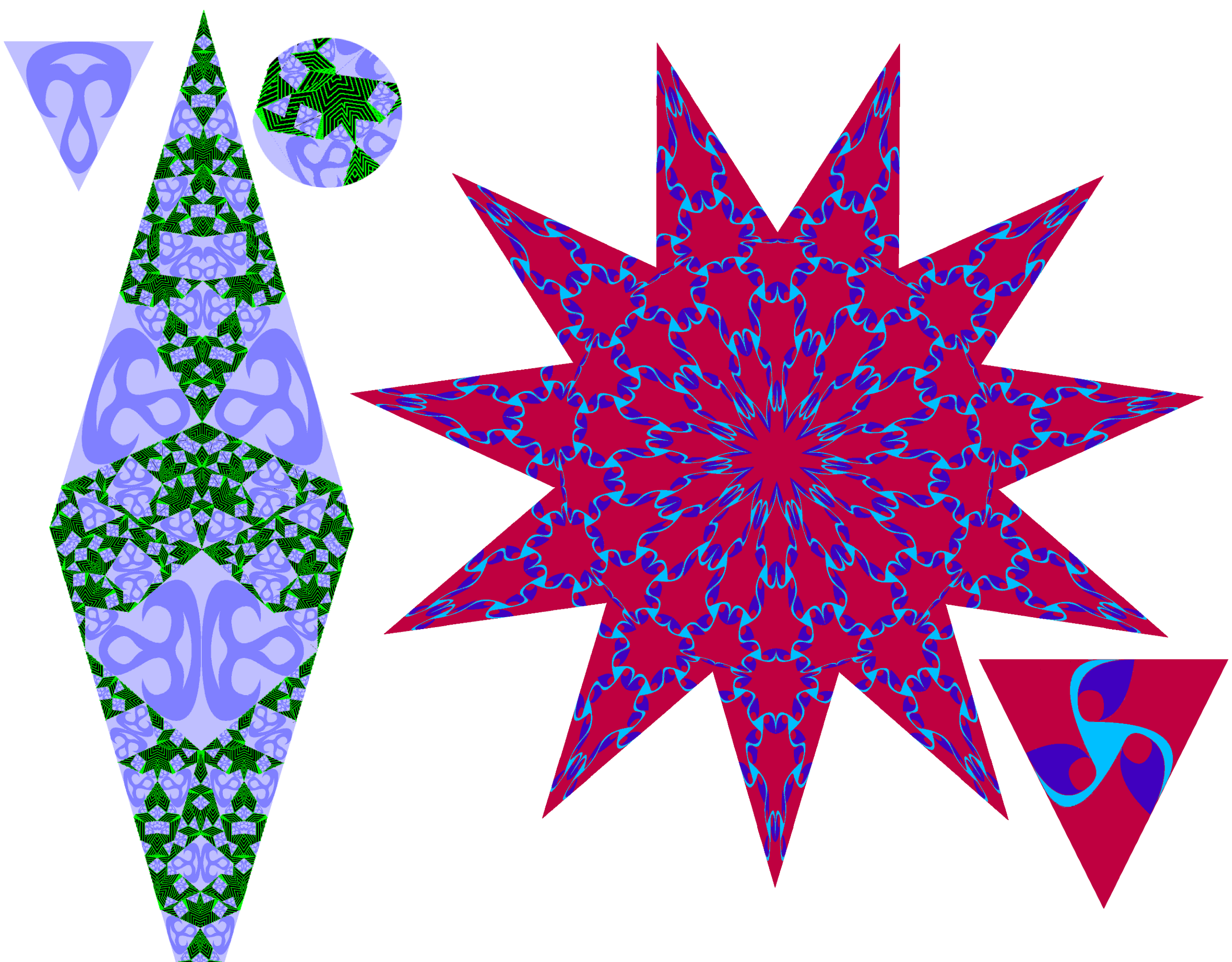


Figure 6: *The rhombus made from two I[1,1,9] triangles. Eleven similar rhombi have been formed into a star. They have been decorated with some old Celtic designs.*

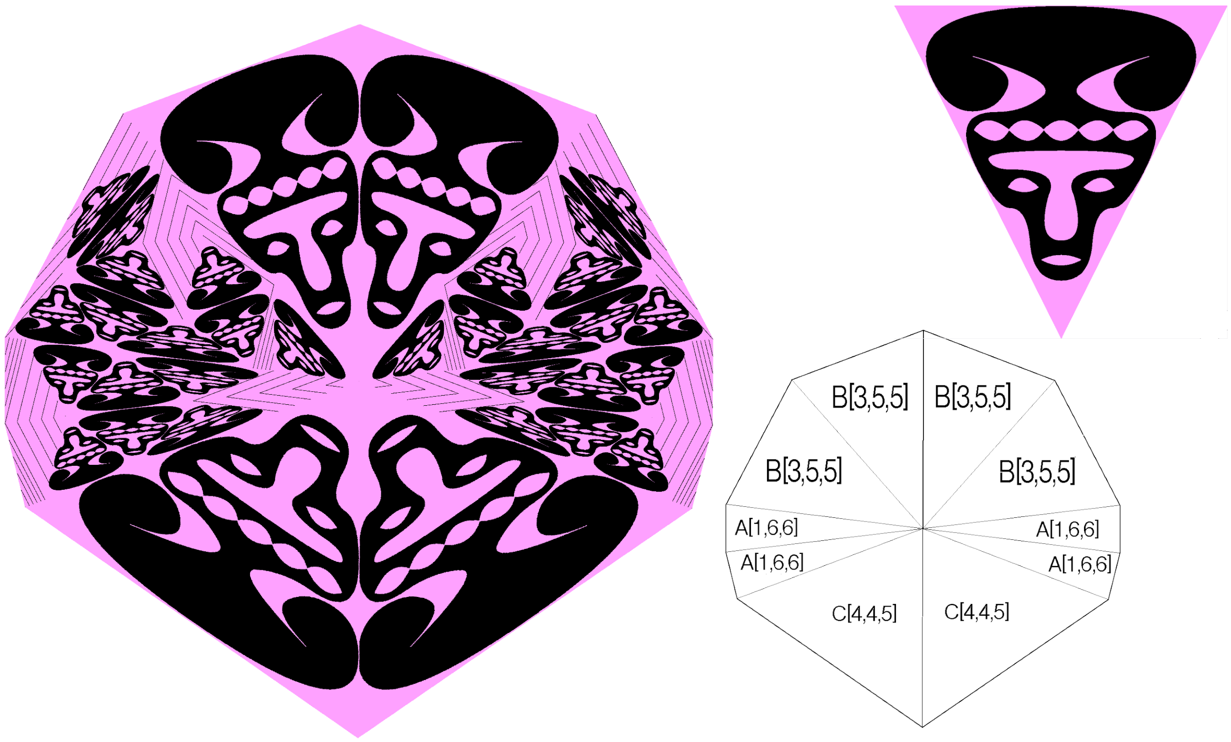


Figure 7: *This is a cyclic polygon because each of the vertices lie on a circle. It is formed using the primary isosceles triangles of the 13-gon. It is decorated by my sketch of a motif taken from an early Celtic bucket.*

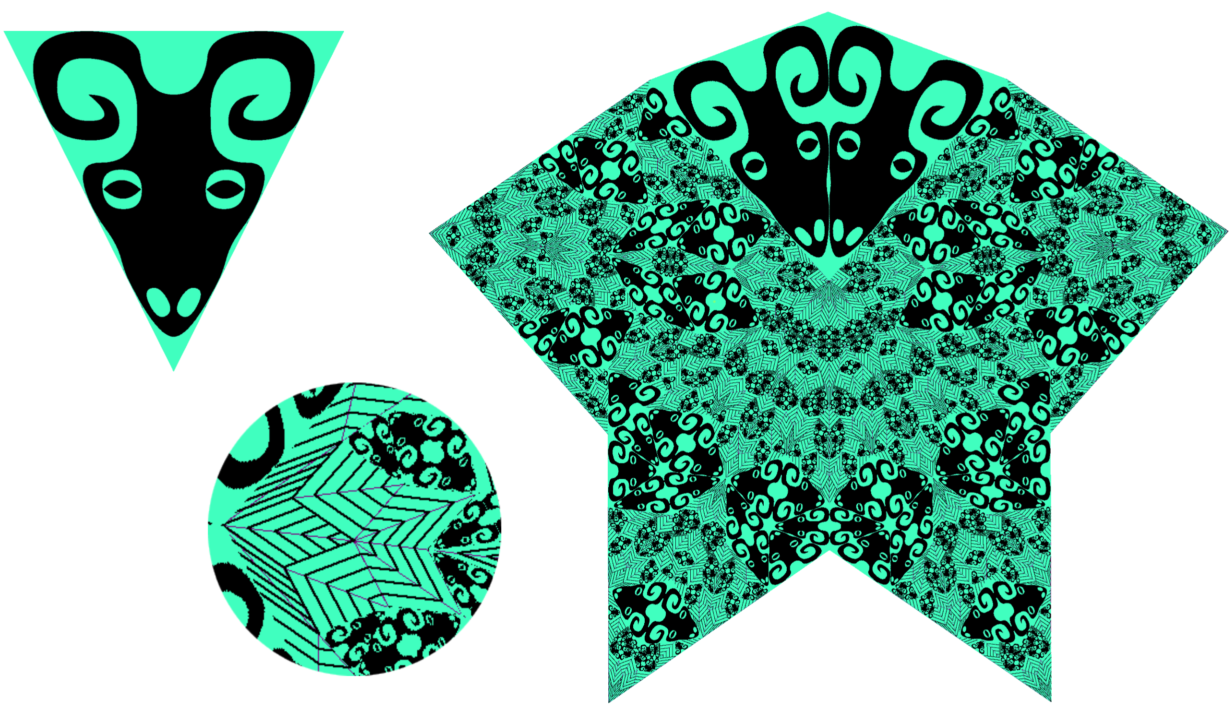


Figure 8: *This is a non-cyclic polygon, because the vertices do not lie on a circle. It is formed using the primary isosceles triangles of the 13-gon. It is decorated by my sketch of a ram motif taken from an early Celtic artefact.*

5. Additional Properties Displayed by Regular Polygons when n is odd

When *n* is odd a regular polygon will display all the properties discussed in section 4. There are, in addition, properties that are displayed only by n-gons where *n* is an odd number. The following illustrations use the algorithm based on the 13-gon. When *n* is odd it is always possible to have an isosceles triangle with frangles *1, (n-1)/2, (n-1)/2* For *(n-1)/2* to be a whole number then n needs to be odd. Starting with this triangle it is possible to construct a 2n-gon and a star with 2n points examples are shown in figure 9.

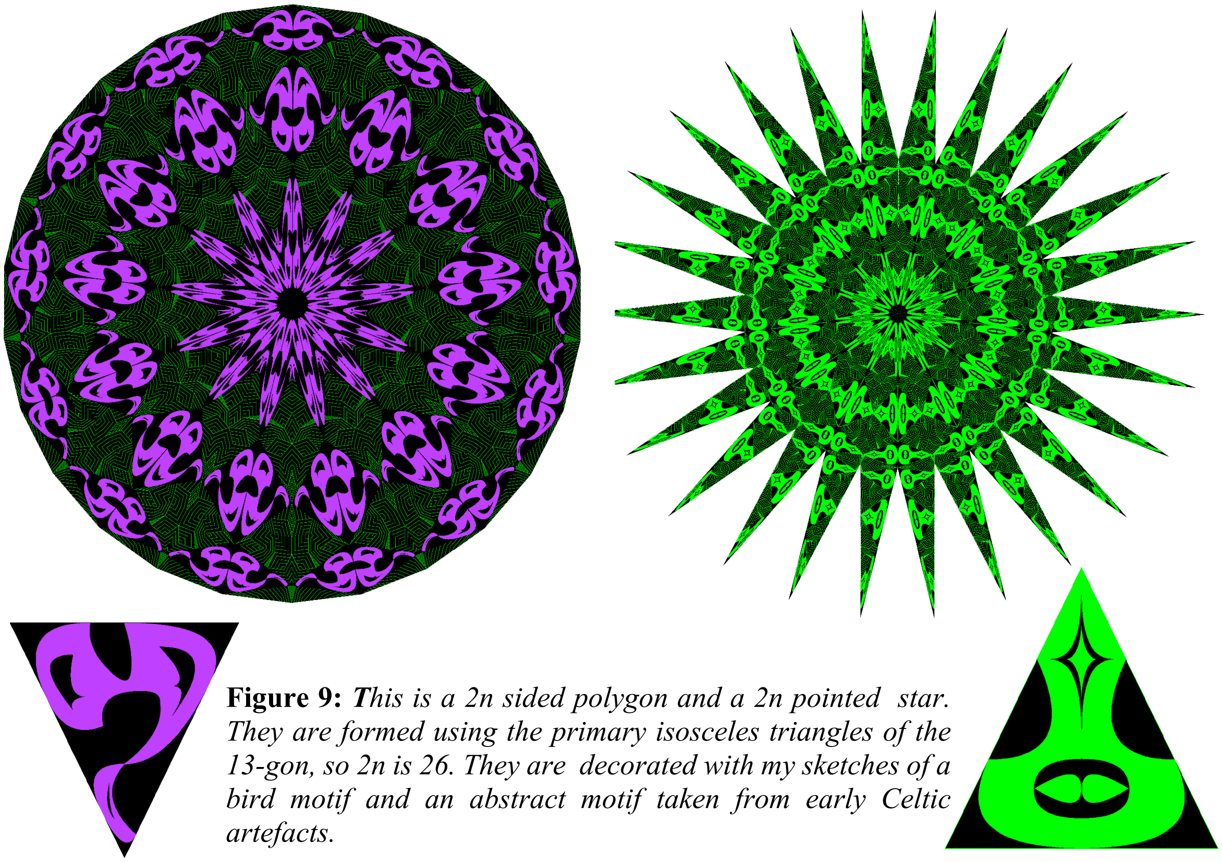


Figure 9: ***T****his is a 2n sided polygon and a 2n pointed star. They are formed using the primary isosceles triangles of the 13-gon, so 2n is 26. They are decorated with my sketches of a bird motif and an abstract motif taken from early Celtic artefacts.*

6. Additional Properties Displayed by Regular Polygons when n is even

When *n* is even a regular polygon will display all the properties discussed in section 4. There are, in addition, properties that are displayed only by n-gons where *n* is an even number. The following illustrations use the algorithm based on the 12-gon, see figure 10. When *n* is even it is always possible to create a shape that is rectangular. This works because the angle for a right angle is *n/2* and when *n* is even *n/2* is an integer. Figure 11 illustrates a rectangular design decorated with a design incorporating the motif of a bull including a spiral which was often used by the Celts. Other rectangles can be made from triangles *B[2,2,8], D[4,4,4]* and a square from 4 of *C[3,3,6]*

7. Additional Properties Displayed by Regular Polygons when n is a multiple of a whole number

When *n* is multiple of an integer larger than 2., a regular polygon will display all the properties discussed in section 4. As well as being even, 12 is also a multiple of 3, 4 and 6. You will notice from figure 10 that the 12-gon has inherent symmetry, unlike the 11-gon in figure 3. Since *12=3x4, 12* is a multiple of 3 The

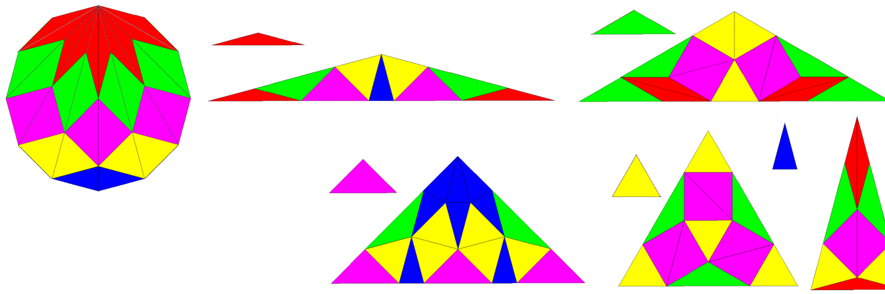
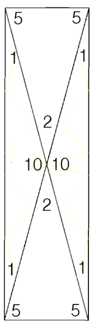


Figure 10: *This shows the dissection of the 12-gon into its primary isosceles triangles. The similar triangles are shown which displays an inherent symmetry.*



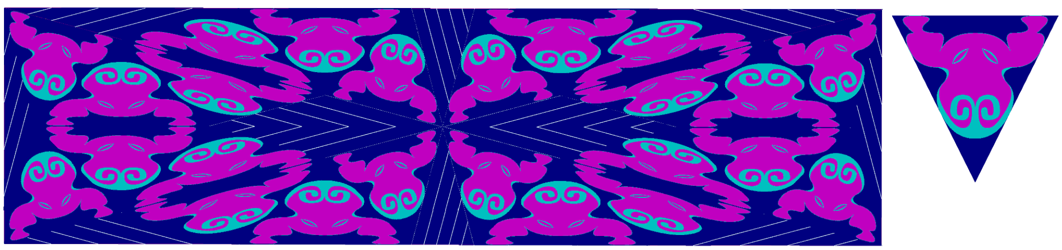
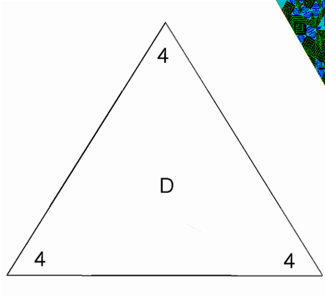
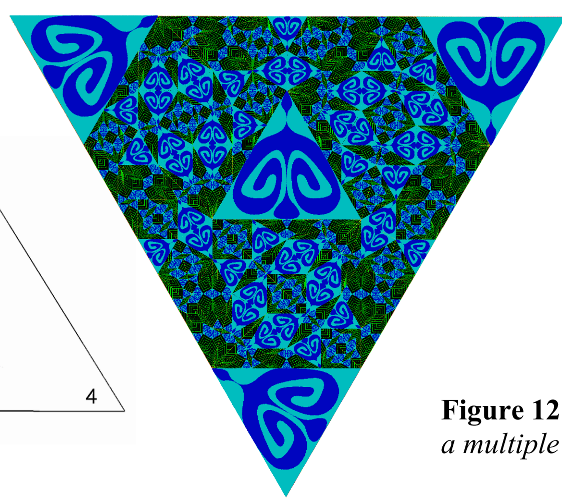


Figure 11: ***T**his shows an arrangement of 4 triangles forming a rectangle for the 12-gon. My sketch of the motif of a Celtic bull has been used as decoration.*

only triangle formed from the 3-gon is the equilateral triangle *A[1,1,1*]. The same triangle can be formed from the 12-gon. Each angle being multiplied by 4 giving *D[4,4,4]*. This can be seen in figure 10. This is possible with any multiple of 3. For example, the 30-gon would have angles *F[10,10,10*] for an equilateral triangle. The Figure 12 shows the triangle *D[4,4,4]* decorated with a spiral motif.The 4-gon i.e., the square can be divided into two isosceles triangle with angles *A[1,1,2]*, this is an isosceles right angled triangle. The angles are n/4, n/4 and n/2. For these to be integral then *n* has to be a multiple of 4. This is true for the 12-gon and the right angled isosceles triangle angles would be C[3,3,6] This can be seen in figure 10. Two of these with a common hypotenuse would form a square as would 4 of them with the hypotenuse as the side of the square. Figure 13 shows a picture formed by the first of these and decorated with a motif I have called the “Grumpy Celt”! One of the isosceles triangles from the 6-gon is *A[1,1,4]*. Since *12=2x6* the same triangle *B[2,2,8]* exists for the 12-gon. Two such triangles have been used to create the design as shown in figure 14 which depicts a drawing of a Celtic Cow motif.





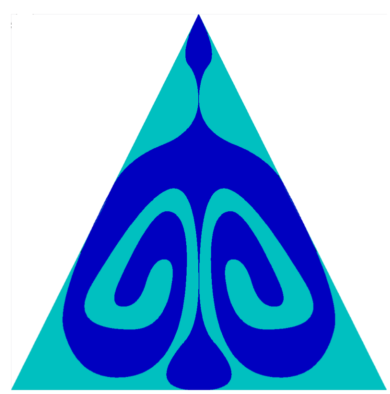
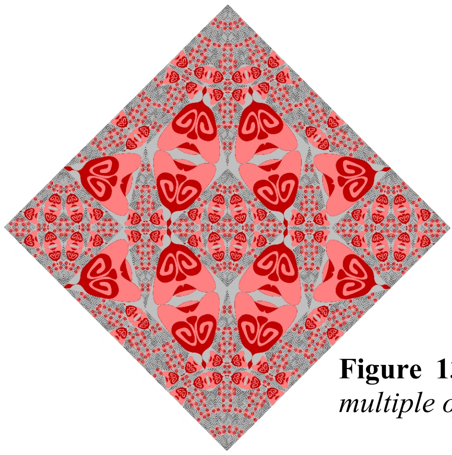


Figure 12: ***T**his shows a triangle D[4,4,4] which is a multiple of A[1,1,1] from the 3-gon*



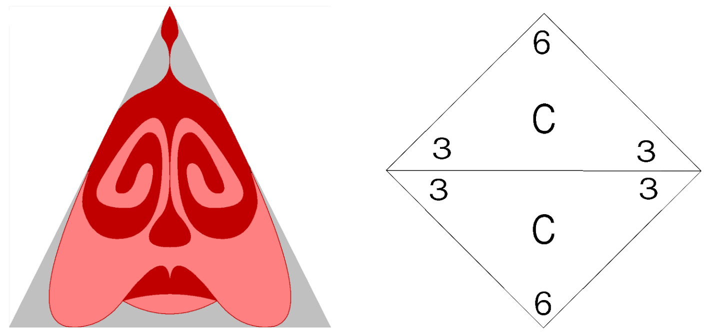


Figure 13: ***T****his shows a square made from two C[3,3,6] which is a multiple of A[1,1,2] from the 4-gon*

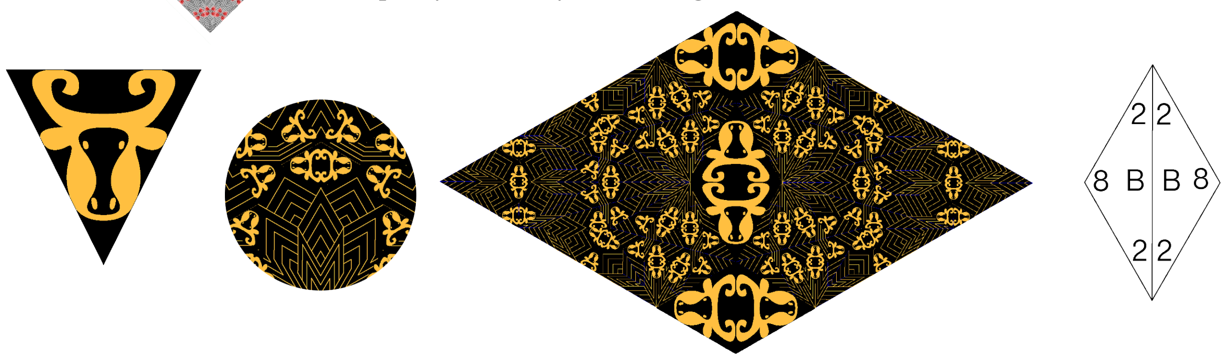


Figure 14: ***T****his shows a rhombus made from two B[2,2,8] which is a multiple from the 6-gon of A[1,1,4]*

8. Conclusion

Although I have shown only examples based on the 11-gon, 12-gon and 13-gon., I am about half way to my ultimate goal of getting to the 36-gon. This will allow me to explore other combinations of polygons such as multiples of 5, 7 etc. I would also like to see what unique properties apply where *n* is a square number. I have tried to consider the inherent symmetries that exist because of the value of *n*. It is often possible to introduce other symmetry into a design. For example figures 13 and 14 display some symmetry because they are based on the 12-gon. I have also introduced a line of symmetry because there are two triangles, one left handed and one right handed.

References

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