The mathematical art of juggling: using mathematics to predict, describe and create

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Abstract

Mathematics has the power to describe, predict and create patterns, a power that is very well demonstrated in the pattern-rich world of juggling. In this paper we examine a simple method of describing juggling patterns using mathematical notation, and then use this notation to predict new juggling patterns. We conclude with a demonstration of how mathematics has been used to create beautiful patterns that did not exist before these mathematical methods had been used, and how mathematical names are now used by jugglers worldwide – a powerful demonstration of mathematics advising the arts.

Introduction

Figure 1: *Nothing but Jugglers*; a juggler juggling jugglers juggling jugglers... (image by author)

Mathematics is the science of patterns, and nowhere can we see that better than in the field of mathematical art. By recognizing, extending and creating patterns we can create artistic representations of stunning beauty, either visually as in prints and sculpture, acoustically as in music or soundscapes, or dynamically as in dance or performance pieces. One of the most powerful examples of this interplay between mathematics and art can be demonstrated in the pattern-rich world of juggling.

The power of mathematics lies in that mathematical tools and thinking enable us to observe phenomena and use these observations to *describe*, *predict* and *create*. This power is well demonstrated in the world of juggling, where mathematics can be and is used to describe juggling patterns, predict which sequences of notation describe viable patterns, and then use notation to create new and beautiful patterns that have never existed before. Perhaps most surprising is that the mathematic notation and names of some of these patterns have entered the common language of juggling and are used by non-mathematician jugglers worldwide.

Figure 1 lends a fanciful introduction into our mathematical exploration of juggling. Titled *Nothing but Jugglers*, it displays the inevitable result of creating a scene with nothing but jugglers. In order to properly represent jugglers, the people in the scene must be juggling *something*, but what can those objects be if the only objects allowed are jugglers? Mathematicians are quick to conclude the scene must be recursive with the number of jugglers growing exponentially to infinity as their size decreases to nothing. (An animated version is available at www.mike-naylor.com).

Mathematicians often find juggling appealing, perhaps because juggling is rich in patterns and combinations. In fact it was computer programmers who first discovered some of the mathematics of juggling in their efforts to create juggling pattern simulators. The first notation was created by Paul Klimek in Santa Cruz in 1981 and further developed by others in the following years [1]. Many juggling notation systems have been created since then, including systems involving multiple jugglers, but none match the elegance and simplicity of the original system known as "Vanilla Siteswap Notation," or more simply as "siteswap."

Vanilla siteswap describes juggling patterns using only a sequence of integers that signifies the length of time each object is in the air. In performance, a juggler may throw objects behind the back, under the leg, or from the back of the hands or elbows. Items may be bounced off the forehead or knees, thrown with palms upwards or downwards, bounced off the floor, or rolled across the chest. Multiple objects may be caught and thrown with one hand, and hands may throw asynchronously or simultaneously. Siteswap notation ignores these tricks, making the assumption that each hand throws just one object at a time, alternately in a simple arc. Despite this severe limitation, this system describes a surprisingly large number of juggling patterns and has been used to create new juggling moves that are now enjoyed by jugglers worldwide.

Encoding throws and basic patterns

A juggler engaged in a repeating pattern moves to the steady beat of the throws and the steady rhythm of the balls slapping into the juggler's hands like a metronome. In siteswap notation the unit of measurement is this "beat"; each beat corresponding to a point in time in which a throw can be made. Left and right hands alternate, so that on the first beat, the left hand throws, on the second beat, the right hand throws, and so on. Each throw in a juggling pattern is labeled by the number of beats which transpire until the thrown object is thrown again. In practice there are about 3 or 4 beats per second.

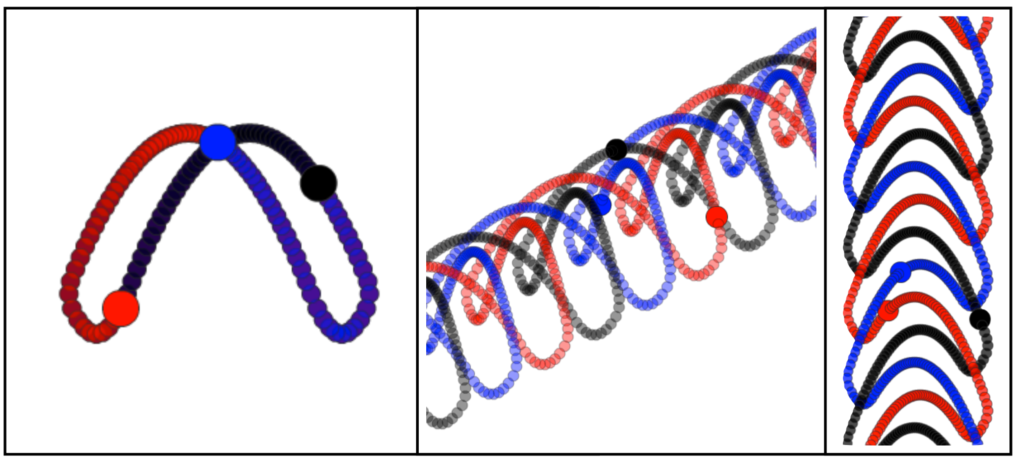


Figure 2: Three-ball cascade with path visualizations

The basic three-ball juggling pattern is known as *the cascade* because it evokes images of water tumbling over itself (many juggling patterns are given names inspired by water-images). In the three-ball cascade, the balls travels in arcs of identical height, from right hand to left and left to right. Since the juggling takes place in a plane parallel to the front of the juggler, we could represent the entire pattern, including the element of time, by stacking snapshots of the location of the balls to create a three-dimensional (four-dimensional?) woven structure as shown in Figure 2. (An interactive animation of this structure can be found at www.mike-naylor.com). A simplified version of a type which we will use to analyze juggling pattern is shown in Figure 3. Each string of arcs represents the path of one ball through time as it travels from hand to hand. Every throw lands three places forward of where it originates in this diagram; each of these places corresponds to a beat. In this case each throw has a duration of three beats. If we were to count "1, 2, 3, 1, 2, 3, 1, 2, 3..." in time to the beats as this pattern were juggled, the same ball would be thrown on the same numbered count every time.

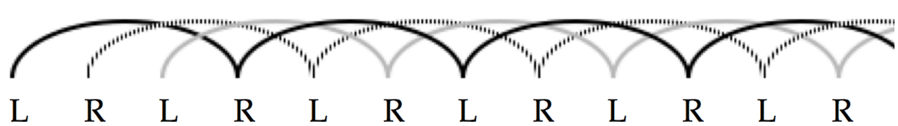


Figure 3: Simplified three-ball cascade space-time diagram

We can encode this pattern by writing down the number of beats of each throw: "3-3-3-3-3-3- ..." By adopting a rule similar to repeating decimals and designating the pattern solely in terms of its repeating digits, the above pattern can be simplified to a single digit: "3."

Figure 4 shows the pattern and the space-time diagram for the basic four ball juggling pattern. Every throw is a “4,” and the pattern name in siteswap notation is therefore also written as “4.” Although the paths appear to be crossing in this diagram, the balls do not cross paths at all – every ball thrown from the left hand returns to the left hand and likewise for the right hand. Because of the parity of this system, even numbered throws return to the same hand while odd numbered throws change hands. This pattern is known to jugglers as “the four-ball fountain.” (The basic patterns with an even number of objects are called fountains because the balls seem to spout like water from the center of the pattern and fall outwards.)

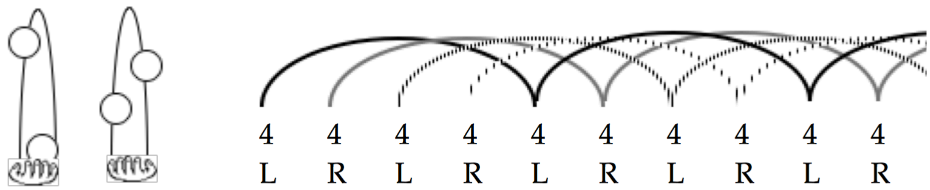


Figure 4: Four-ball fountain and its space-time diagram

In general, a one-digit pattern notation denotes the basic juggling pattern for the same number of objects. A “5” is the 5-ball cascade, a “6” is the 6-ball fountain, etc. (See Table 1). Consider yourself lucky if you get the chance to witness a “7,” “8,” or “9.” Very few jugglers have mastered the 7-ball cascade, while perhaps less than a dozen can juggle nine or more.

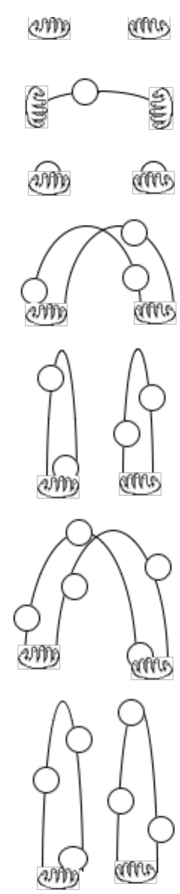
#	**as a throw**	**as a pattern**	
0	empty Hand	both hands empty	
1	fast hand-off	one ball shuttled back and forth between	
2	hold	one in each hand, no motion	
3	chin-level arc from one hand to other	three-ball cascade	
4	head level throw returning to same hand	four-ball fountain	
5	above head-level throw from one hand to the other	five-ball cascade	
6	high throw returning to same hand	six-ball fountain	
7	. . . etc . . .		

Table 1: Digits, throws and patterns

Throws corresponding to digits less than 3 warrant some explanation. A "0" is used to denote an empty hand for one beat. A "1" is a quick hand-off to the other hand. A "2" would, in theory, leave the hand with a very short toss, and be caught immediately with that hand taking no action between throw and catch. In practice, however, this is almost never done. Instead, a 2 designates a pause; the hand holds the ball for that beat without throwing it. The pattern designated by the single digit "2," then, would be right-hand pause, left-hand pause, right-hand pause, left-hand pause ... in other words, this pattern is simply holding one ball in each hand and not doing anything – a juggling pattern most everyone can do!

Two-Digit Patterns

Showers. Most non-jugglers believe that the proper way to juggle three balls is in a "circle," one hand throwing the balls in a high arc, the other hand catching and passing the balls quickly to the throwing hand. This pattern is called "the shower" (another water name!) and it is actually quite a bit more difficult to perform than the cascade. The space-time diagram for the three ball shower is shown in Figure 5.

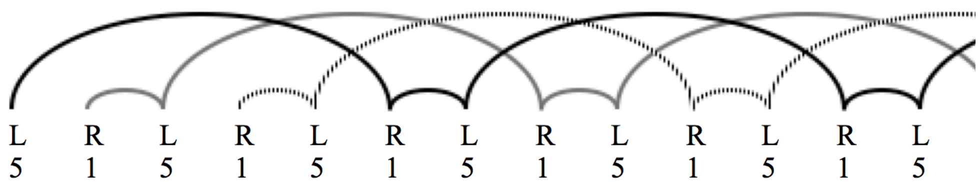


Figure 5: Space-time diagram for the three-ball shower

In this case, the left hand is doing the high throws, the right hand is quickly shuttling each ball to the left hand where it is immediately thrown again. The balls that leave the left hand land five beats later, and the balls that leave the right hand land one beat later. This patterns requires two digits to encode as there are two different types of throws being made. It is encoded as the repeating unit of digits, that is, "5-1."

By convention, we usually name the pattern with the greatest digit first, although this is not necessary. One could easily say that a "1-5" is the same pattern as a "5-1" but in the opposite direction. Owing to the ambidextrous nature of juggling and the arbitrary choice of which hand to start the pattern with, we will not make a distinction and instead refer to patterns and their reflections by the same name.

By increasing the height of the high throw, we "make room" for additional balls. A four-ball shower would be a "7-1," a five-ball shower is a "9-1," and so on. You may wish to verify these with space-time diagrams. The record for objects juggled in a shower pattern is eight – an incredible "15-1" pattern performed by a woman on the island of Tonga in the Pacific [3]. In Tonga, juggling is an activity for girls, many of whom juggle limes or tui-tui nuts. When Western jugglers first came to Tonga, islanders thought it was very funny to see men juggling!

Half-Showers. Another class of juggling patterns are the half-showers. They are similar to showers in that one hand throws the balls over the top of the pattern, but instead of the other hand making a quick hand-off, the ball is sent across in short arc. Half-showers are somewhat slower and easier to perform than showers.

The diagram for the four-ball half-shower is shown in Figure 6. Here, the left hand always makes high a throw to the right hand, which returns the ball to the left in a shorter arc. This pattern is a "5-3."

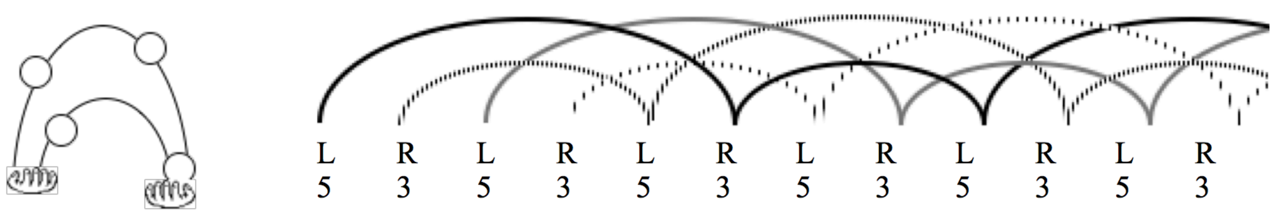


Figure 6: Four-ball half-shower

One-Hand Patterns. Two balls in two hands is just not challenging enough for most jugglers; when we speak of a two-ball juggle, we generally mean two balls in one hand as shown in Figure 7.

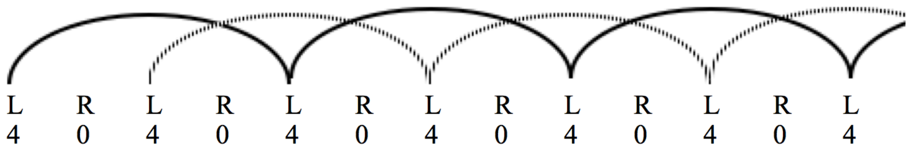


Figure 7: Two balls, one hand

The zero denotes the empty hand; this pattern is encoded as "4-0." Notice how similar the path diagram is to the four-ball fountain. In effect, the paths of the two balls in the right hand have been removed, and a zero entered as a place-holder. Similarly, three balls in one hand (an impressive feat!) is encoded "6-0," and four balls in one hand is an "8-0."

Suppose we were juggling two balls in one hand as above (the "4-0"), and picked up a ball in the other hand and simply held it. This three-ball pattern would be no more difficult than the "4-0," but it would have a different siteswap name: a "4-2" (see Figure 8). Likewise, juggling three in one hand and holding a fourth in the other hand would be designated "6-2."

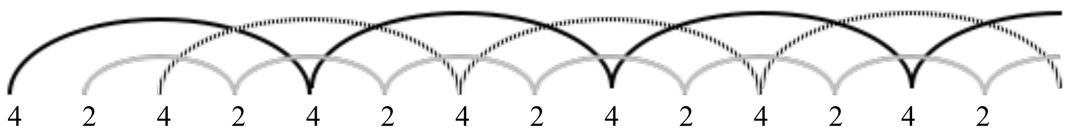


Figure 8: Two in one with the other hand just holding a ball

Patterns of Patterns. The two-digit patterns have been organized in Table 2, and the "patterns of patterns" within the table have been extended both downwards and to the right to include such patterns as the "6-4," a five-ball pattern in which one hand juggles three by itself, the other hand juggles two.

	1 hand	shower	1 hand + hold	1/2 shower	1 hand + 2 in 1	high shower	1 hand + 3 in 1	super shower	1 hand + 4 in 1
1 ball:	2-0								
2 ball:	4-0	3-1							
3 ball:	6-0	5-1	4-2						
4 ball:	8-0	7-1	6-2	5-3					
5 ball:	10-0	9-1	8-2	7-3	6-4				
6 ball:	12-0	11-1	10-2	9-3	8-4	7-5			
7 ball:	14-0	13-1	12-2	11-3	10-4	9-5	8-6		
8 ball:	16-0	15-1	14-2	13-3	12-4	11-5	10-6	9-7	
9 ball:	18-0	17-1	16-2	15-3	14-4	13-5	12-6	11-7	10-8

Table 2: Two-Digit Patterns

This listing quickly extends beyond the capabilities of human jugglers, but it accommodates all possible two-digit siteswap patterns. Notice that the sums of the digits of each of the two-digit patterns is twice the number of objects being juggled – a property of these patterns we will generalize shortly.

Three Digit Patterns

A “quick and dirty” way to generate some three-digit patterns is to remove one or two paths from the three-ball cascade. Removing one ball from the pattern gives us the two-ball pattern “3-3-0” (see Figure 9).

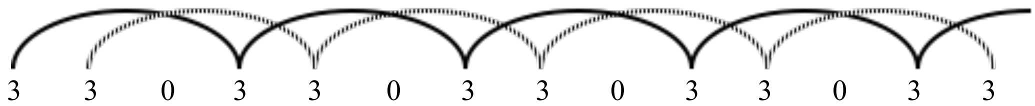


Figure 9: Two balls in three-ball cascade pattern

Removing one more path reduces it to a one-ball pattern: “3-0-0” (see Figure 10).

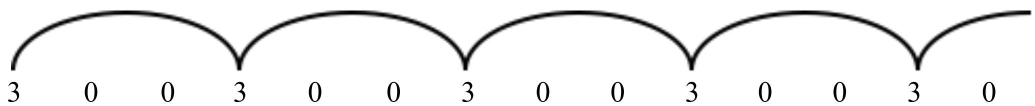


Figure 10: A very simple 3-digit pattern

Notice that the “3-3-0” is a two-ball pattern with a digit sum of 6, and the 3-0-0 is a one-ball pattern with a digit sum of 3 – both sums are three times the number of objects being juggled. Incidentally, the “3-3-0” and the “3-0-0” are excellent patterns to practice when one is learning the 3-ball cascade.

Let’s replace the zero in the “3-3-0” with a different digit. This digit must represent a throw that will land in an empty hand, so throws of 1, 2, 4 or 5 will not work; these throws would result in two balls landing in a hand. A six, however will work - six spaces to the right of a zero in the space-time diagram is another zero which represents an empty hand and thus a potential landing spot for this additional object. Changing the “0” to a “6” actually creates two new paths in the pattern, since each new set of arcs connects every other empty space in the pattern (see Figure 11). The “3-3-6” or equivalently “6-3-3”, is a four-ball pattern wherein two balls cascade back and forth between hands while the other two balls stay one in each hand in alternating high throws. This pattern has a digit sum of 12, again, 3 times the number of objects. If instead of changing the zero to a six, we change it to a nine, there will be room for an additional object in the pattern. A 9-3-3 is a five-ball pattern, and the digit sum is exactly three times the number of objects.

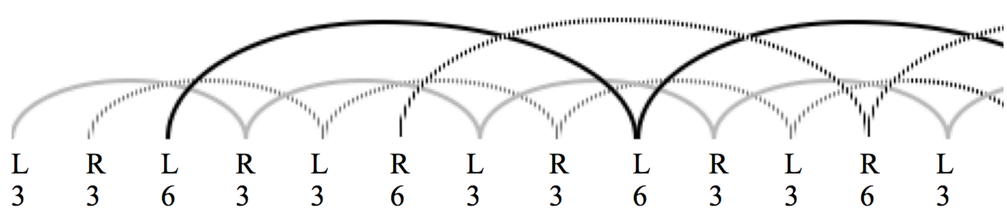


Figure 11: The four-ball 6-3-3 pattern

Properties of Legal Patterns

As we have seen, one digit patterns are equal to the number of objects juggled, in two digit patterns the digit sum is twice the number of objects in the pattern, and in three-digit patterns the digit sum is thrice the number of objects being juggled. In fact, *the mean of the digits in a siteswap pattern is exactly the number of balls in the pattern*. If the mean of the digits is not an integer, the pattern will not work. A 2-1, for instance, tells the juggler to hold a ball in the left hand, make a quick pass from the right, hold the left, quick pass from the right ... the right hand never receives any balls, and the left hand never gets rid of any balls, an impossible situation.

The following proof that the mean of the digits equals the number of objects being juggled is based on a clever argument by Warren Nichols of Florida State University [4]:

Assume we have a legal pattern of n digits and a digit sum of m. In the example below we consider the legal pattern 6-3-0-2-4; this pattern has 5 digits and a digit sum of 15. Arrange these digits in a circle and construct a space-time diagram which, instead of being linear, wraps around the circle to join itself in a continuous manner (Figure 12). Since this is a legal pattern, the paths are uniformly distributed around the circle, that is, there could not be a place in the space-time diagram where there were, say 3 paths, and another place where there were 4 paths, as this would indicate that the number of objects being juggled changes mid-pattern. The circumference of this circle is n beats since there are n digits, and the total length of the paths is the sum of the digits or m beats, so the paths wrap around the circle m/n times. The paths are uniformly distributed, so at every point in time during this pattern there are m/n paths, which means there are m/n objects being juggled.

The wrapped space-time diagram for 6-3-0-2-4 appears below. The circumference is 5 beats, the total length of the paths is 15, indicating that the paths must wrap around the circle 15/5 = 3 times. 6-3-0-2-4 is thus a three-ball pattern.

Conversely, if m/n is not an integer, then the paths cannot wrap around the circle an integral number of times. The number of objects being juggled must change during the pattern, and therefore the pattern is not legal.

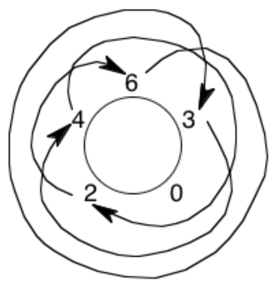


Figure 12: The total length of the paths is 15 and the circumference is 5; the paths wrap around the circle 15/5 = 3 times.

The fact that the mean of the digits must be an integer is a necessary but not sufficient condition for the digits to represent a legal pattern. One can create strings of digits which average to an integer, but which are impossible to juggle.

5-4-3, for example, has an integral average of 4. The space-time diagram in Figure 13 shows what happens when this pattern is attempted.

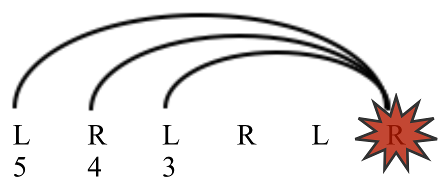


Figure 13: Collision!

All three balls land at the same time in the same hand! While in real-life, this would be an impressive feat, in siteswap notation this situation is referred to a collision, and it is not allowed. In order for a pattern to be legal, only one object may land in a hand at one time.

The problem with this pattern is not the choice of digits, but rather the order of the digits. If we permute the digits as "5-3-4," the pattern works just fine as shown in Figure 14.

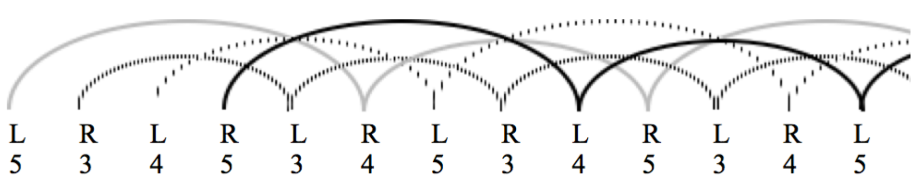


Figure 14: Collision fixed

Construction of a space-time diagram is not necessary to determine if a pattern is legal. Other mathematical tools can be found in a paper included in the references [4].

The Lost 3-Ball Pattern and Other Patterns

We now have all the tools necessary to create and test legal patterns. All that is left it to use these tools to creatively explore the space of possible juggling patterns. One of the truly great powers of mathematics is exemplified in the story of siteswap notation. As jugglers studied pattern notation, they discovered patterns that were beautiful and unknown. One of the most striking is the “4-4-1” pattern (Figure 15). This pattern consists of two staggered throws straight up, then a quick hand-off which is immediately sent up as the first of a different pair of staggered fours (Figure 18). This pattern is beautiful to see and satisfying to perform, and was not discovered until mathematics showed the way!

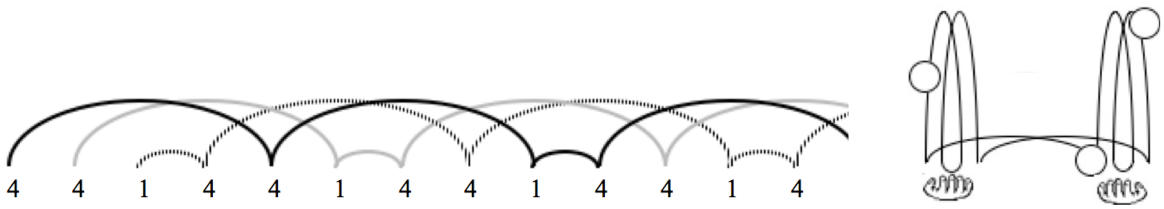


Figure 15: The 4-4-1 (“The lost 3-ball pattern”) is referred to by jugglers worldwide as the “4-4-1”.

While many juggling patterns have fanciful names such as the cascade, the shower, and the fountain, in juggling circles the “4-4-1” pattern is called by its siteswap name. The mathematical terms have entered into juggling culture worldwide – a powerful testament to the interplay between mathematics and the arts. Other siteswap pattern names broadly used by jugglers include the 5-3-1 and the 5-5-5-0-0. Juggling is one of the few arts in which practitioners regularly use names derived from mathematical notation!

These pattern names exemplify the true power of mathematics. We can observe a phenomenon and create mathematical notation to describe it, look for patterns and generate rules within our notation, and then use these rules to create new symbolic patterns which correspond to new discoveries within the natural world. Not only does mathematics have the power to describe – *mathematics has the power to create!*

Finale

Several siteswap juggling pattern simulators are available for free on the internet, but siteswap notation is more than just a powerful tool for creating electronic jugglers. Siteswap can aid jugglers while practicing complicated patterns by focusing attention on individual throws, and the notation has been used to discover new patterns which were previously unknown anywhere. Incidentally, there are 11 legal three-digit three-ball patterns – finding them all and visualizing what they look like is a nice challenge.

Regardless of whether or not a juggler understands or uses siteswap, juggling is still a sport of kinesthetic mathematics. One cannot help but be mystified and drawn into the intricate web of translations, inversions, dilations, and permutations as a skilled juggler weaves patterns upon patterns of order balancing on the edge of chaos.

References

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[2] Knutson, A., *Site-Swap FAQ*, [Online] Available: http://www.juggling.org/help/siteswap/faq.html [January 12, 2012].

[3] Truzzi, M., *On Keeping Things Up in the Air*, Natural History, 88, 10, December 1979.

[4] Naylor, M., *Patterns of patterns: the mathematics of juggling,* The Journal of Recreational Mathematics, 32, 1, 2004.