# Sacred Numbers and Tessellations in Aquileia's XI Century Mosaics

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## Abstract

Many religious and/or esoteric traditions attribute specific meanings to small integers. In particular, in Christianity, the number 7 symbolizes the Holy Spirit, the number 8 the union of the human body with the spiritual world, and the number 33 is Christ's (alleged) age at crucifixion. A mosaic pattern found on the floor of a XI century cathedral in northern Italy closely covers the interior of a regular octagon with a rotationally symmetric, 7x7 checkerboard array of squares. When one considers the largest possible set of sub-squares inside that figure's 8 sides that forms a cross-like shape, the number 33 coincidentally manifests itself. We argue that the ancient mosaic designers chose the 7x7 subdivision because they recognized this hidden count of 33, rather than because no other low-resolution subdivision, with the possible exception of the 10x10, would have worked as well mathematically (without, however, generating directly the numbers 8 and 33).

## Introduction

As is well known, all esoteric and religious traditions attribute small integers a specific meaning that usually depends on their specific speculations; Christianity, in particular, attributes, among others, a special and central meaning to the numbers 8 and 33. In this paper we discuss a tessellated mosaic in the Christian Basilica of Aquileia and we show how a clever use of the (fairly well known) mathematical properties and proportions of octagons allowed mosaic makers in the XI Century to obtain a shape in which the numbers 8 and 33, together with the Christian symbol of the Cross, appear either explicitly or implicitly. This construction served to incorporate these symbols into part of the floor. The only way to obtain such numbers in an octagon was to subdivide it into a 7x7 array of squares; any other subdivision would, of course, produce different numbers. We are indebted to the referee for having pointed out to us that a similar emotional and esoteric significance could be obtained also by means of a 10x10 array that produces the number 64 (i.e., the square of 8) but unfortunately not the 33. We may argue that the mosaic makers did choose the 7x7 simplest array because the 10x10 array would produce much smaller squares (and moreover it would not easily include the more symbolic 4x4 sub-array used, since it would be based on squares with an odd rather than even number of subdivisions) – see Figure 3 later in the paper.

## Number and Symbols in the Cathedral of Aquileia

***Aquileia*** was an ancient Roman own (and, in fact, the most important harbor of the Northern Provinces of the whole Empire) in that part of Eastern-Northern Italy that is now called *"Regione Friuli Venezia-Giulia"* (from the name of the Roman Emperor *Julius Caesar*). The current Italian name comes from the Latin word *Aquila* (*Eagle*) that was in fact the symbol of the town. After the fall of the Roman Empire in V Century AD the town of Aquileia maintained its importance as one of the many flourishing centers of the Eastern Roman Empire and it became in fact an important centre of Christianity. Aquileia is situated along the Adriatic Sea (at the edge of the "Lagoons" that include the most famous town of Venezia). Even if the actual village is small (about 3500 inhabitants) it was nevertheless a rather important town in XI, XII and XIII Centuries, when it was the seat of a powerful and rich Archbishop; because of this it is one of the main archeological sites of North-Eastern Italy.

A particular importance is attributed to the mosaics of its Cathedral (a flat-roofed "Basilica" erected by the Patriarch *Poppo* in 1031 on the site of an earlier church). A Romanesque façade brings through a "Portico" to the *Church of the Pagans* (what it remains of the 5th Century Baptistery, that has a circular shape in which, as usual, the "sacred" motives of regular hexagons and octagons dominate). As is well known, the hexagon is related with the number 6 (the "working days of creation") while the octagon is related to the number 8. In early Christianity, this number symbolized many hidden sacred meanings: the number 4 is the "number of the body", so that the "double quaternary" esoterically hidden in the number 8 (i.e., the double of 4) was considered to represent the "union of the human body and the spiritual body", i.e. a way to conjugate the material and the spiritual essence of human beings. The number 4 is in fact the number of *Evangelists* (it appears frequently in the *Apocalypse* by *Giovanni Evangelista*) and it refers also to the *number of elements of matter* (i.e., Earth, Water, Air and Fire) and to the *number of directions in space* (North, West, South and East). See [1] for more details.

What is most important for us is the interior of the Cathedral, where in a rich and noteworthy mosaic pavement (dating back to a period ranging from the 4th to the 11th Century AD) a large number of fantastic and artistically valuable shapes appear to be geometrically inspired – see Fig. 1. We shall limit ourselves to one of them, sitting in the lower part of the left nave of the church, that we found to be particularly and extremely significant because of its geometrical, arithmetical and esoteric characters related with well known properties of octagons.

## Octagons, Tessellations and Silver Rectangles

Many of these mosaics explicitly or sometimes implicitly refer to the joint themes of the (regular) octagon and of the number 8. The masters of the Mosaics of this Cathedral faced the problem of filling such an octagonal-shaped mosaic with black and white tessellations formed by small squares, each one of them being formed by a number of even smaller "tesserae" (incidentally, each one of the small Squares used is formed by 16 smaller "tesserae"). They did it in a way that allowed them to obtain both the "sacred numbers" 8 and 33 out of the octagons used and (even if there is no real evidence of such a fascinating hypothesis) we like to argue that such a choice could have been made on purpose, as a by-product of the "art of mastering numbers" for architectural scopes that had already a long tradition (since the time of Babylonian and Egyptian cultures). As mentioned above the only way to obtain such numbers was to subdivide the octagon by means of a 7x7 grid (or by a less simple 10x10 grid, see Fig. 3 - properties that depend on the proportions of the octagon and, in particular, on "best fit" approximations of the square root of 2 involved in its size). We mention, moreover, that also 7 and 10 had a sacred meaning at the time.

Of course it is not possible to fill perfectly the whole shape of a regular octagon by a chessboard of small black and white squares: some of them have to be necessarily cut to (rectangular) triangles (i.e., half squares) situated along the oblique sides of the octagon. Referring to Fig. 2, we recall that the octagon can be embedded into a larger square so that, cutting away half of four smaller squares at the four vertices, the octagon turns out to be formed by 8 identical sides, four of them being suitable parts of the sides of the "big square" and the remaining (oblique) four sides being the diagonals of the four smaller squares.

It is convenient to assume that the side of octagon is of length 1, so that the correct (irrational) proportions tell us that, being 1 the length of the octagonal side (and accordingly also of the diagonal of the small squares), the side of the squares cut away is $l_{triangle} = 1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) (by Pythagoras' theorem). The side of the octagon and the side of the square are in fact incommensurable. The side of the big "enveloping" square is therefore $l_{square} = 1 + \sqrt{2}$ ($\approx 2.4142$).

**Figure 1**: *Octagons in the floor mosaics of the Cathedral of Aquileia - © photo by Marcella Giulia Lorenzi*

Cutting away the four small squares, in the interior of the octagon one is left with a Cross (that can be defined as a "truncated Greek cross"). Reverting to the sacred meanings of these symbols, it is first of all important to recall that the black and white tessellation has a well known "esoteric meaning", according to which black and white refer in fact to the eternal duality between darkness and light, between night and day, between evil and good. We also mention that in Christianity the Cross is a fundamental symbol; it is again related to the number 4, as it is characterized by exactly four vertices: the vertical and the horizontal arms of the Cross recall the earth (the "horizontal plane") and the sky (the "vertical line") so to enforce in the Cross the role of "bridge" between mankind and God. This is a further reason why the number 8 is central, since it assumes the double valence of a "double four"; moreover, being mostly related to the notion of "rebirth" and "regeneration" (since, according to the Gospels, a period of exactly 8 days passed between the entrance of Jesus Christ in Jerusalem and his resurrection), the octagon - adding the strength of a very regular geometric shape - is also the standard format of baptismal fountains. A final mention is deserved by the number 33, which in Christianity explicitly refers to the (alleged) age of Christ at the time of his crucifixion. See again [1].

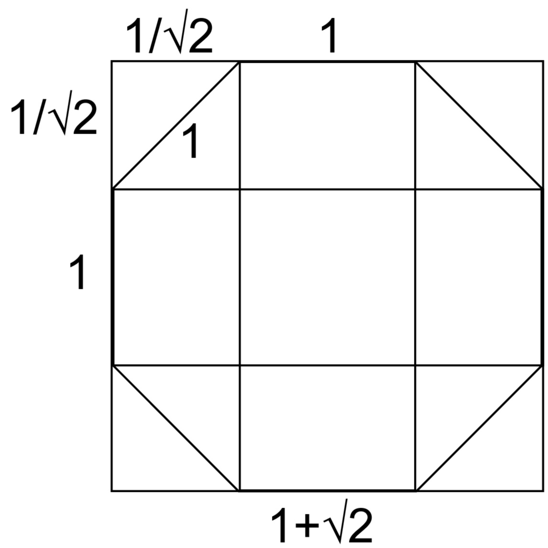


**Figure 2**: *The octagon embedded into the square, the cross and silver rectangles*

Let us now revisit our octagon embedded into the square in a more detailed way. Drawing the parallel lines that join horizontally and vertically the 8 vertices of the octagon, one divides it into exactly 9 pieces: a central square (of side 1), four triangles (that are identical to those that have been cut away at the corners) and four rectangles with sides 1 and $1/\sqrt{2} = \sqrt{2}/2$. Cutting away the four triangles, the remaining 5 pieces form the Cross (see Fig. 2). We remark that also the numbers 5 and 9 have a particular meaning according to Platonic and Pythagoric conception [1],[2]: 5 as the "augmented" number of "fundamental components of Matter" (the four elements plus "Quintessence", reflected in the exact number of "Platonic solids"), while 9 as the "triple triad", where the number 3 is another fundamental number of Christianity (because of its intimate relation with the "mystery of Trinity"). The central Cross is in fact formed by the juxtaposition of two rectangles, one horizontal and one vertical. They are silver rectangles (see [3]), since their measures are 1 and $(1 + \sqrt{2})$. Let us recall that a *silver rectangle* is a rectangle having proportions in the ratio $(1 + \sqrt{2})$, an irrational number also known as *"silver number"*. The silver number is in turn a particular case of so-called *metallic numbers* (see [3] and references quoted therein), i.e. the positive roots of an algebraic equation with integral coefficients

$$x^2 - px - q = 0$$

with positive $p$ and $q$. The metallic number associated to the pair $(p,q)$ will be denoted by *Met(p,q)* and it is given by

$$Met(p,q) = 1/2\ [\ p + \sqrt{(p^2 + 4q)}\ ]$$

The "golden number" $\Phi = Met\,(1,1) = 1/2\ (1 + \sqrt{5})$ is obtained by choosing the pair $(p,q) = (1,1)$, while the silver number corresponds to setting $(p,q) = (2,1)$, so that $Met\,(2,1) = 1/2\ (2 + \sqrt{8}) = 1 + \sqrt{2} \approx$ 2,41421356237… Recall finally that these numbers can be obtained under the form of *continued fractions*; in particular $Met\,(2,1) = 1 + \sqrt{2}$ is the continued fraction denoted by $[\underline{2}]$ (see [3]).

## Analyzing the Tessellation in the Octagonal Mosaic of the Cathedral of Aquileia

Before proceeding in our mathematical analysis, let us now give a look now at the octagon in the mosaic of Aquileia (Fig. 3). One realizes that there are exactly 7 rows and columns of small black and white squares, that by symmetry are separated as two in the left and right columns and three into the central Cross. The questions that we are addressing are: *Why these and not other numbers? Why a subdivision based on 7 lines and rows was chosen?* Notice also that the octagon contains exactly 16 small black squares and 21 white ones, for a total of 37 small squares. However, as we mentioned above, one cannot construct a tessellation of the octagon by just using squares; at the oblique edges one has to cut these squares in two parts and use right-angled (small) isosceles triangles. There are therefore black half-squares along the oblique sides. Counting these halves of black squares the total area covered is formed by adding 8 more triangular pieces, that sum up to 4 more black squares (bringing the total to 20 black and 21 white "mosaic tesserae" and the overall total to 41 "tesserae"). Again one can wonder: *Why these numbers and not other ones…?*

The mathematical reason is very simple: it resides in the "square root of 2" (that, as we know, together with $\Phi$ was the basis for Pythagorean philosophy on *irrational numbers* - see [1],[4]); and, of course, it resides in the practical necessities that were encountered by the master mosaic-makers in constructing this black and white tessellation in the octagon with the Cross in its interior. Further practical reasons were due to the fact that they obviously had to cope with small mosaic "tesserae", having a finite size that could not be reduced at will and (in XI Century mosaics) were usually of the order of 1 centimeter. The mosaic masters had therefore to cope with suitable rational approximations of the sizes of both the octagon and the silver rectangles (related with the irrational $\sqrt{2}$).

Let us analyze the linear measures of these figures and explain why it was necessary (or at least "convenient") to divide the left and right parts into 2 columns, while the central Cross was divided into 3 columns: it is, as we said, a rational approximation of exact irrational numbers, but it was "the best approximation" that could be used by the mosaic masters under the aforementioned circumstances. Let us first notice that $1/\sqrt{2} \approx 0.7071 \approx 0.6666\ldots = 2/3$, so that - with a reasonable approximation - the number of parts to be cut along the side of the octagon has to be (approximately) 3/2 = 1 + 1/2 times larger than the number of subdivisions of the cathetuses of the triangles cut away at vertices. The simplest choice is therefore 2 and 3 even if one could use any other "proportional pair" (e.g., 4 and 6); but with any other of these alternative choices one would of course obtain a number of tesserae much larger and could not fit the numbers 8 and 33 into the design. One should moreover notice that $l_{triangle} = 1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) $\approx$ 7/10 $\approx$ 2/3 (with an error of 7/10 – 2/3 = 1/30); this gives a further reason to dividing the side of the octagon in three parts and the side of the Triangle in two parts (2 + 2 + 3 = 7). Notice also that a rough approximation of the square root of 2 gives $\sqrt{2}$ ($\approx 1.4142$) $\approx$ 1.5 = 3/2, that is in fact the inverse of 2/3 (all fractions that appear in the Pythagorean Musical Scale).

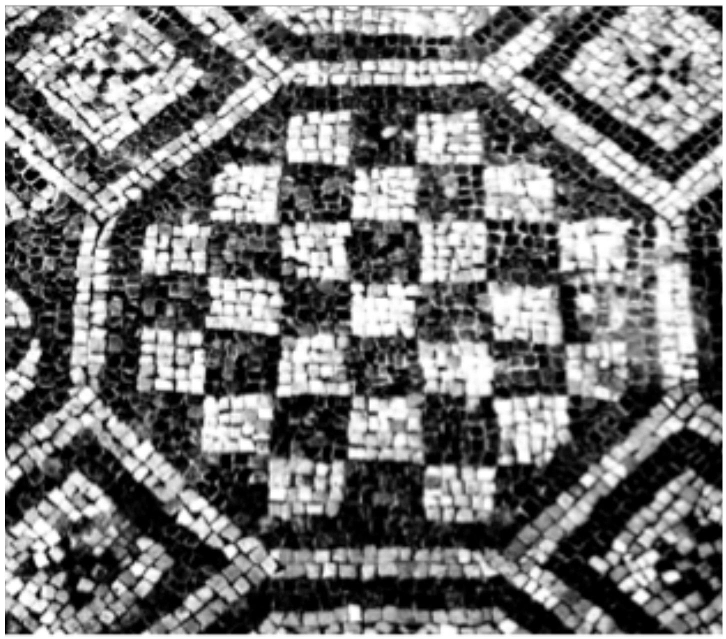

**Figure 3**: *The Tessellation of the octagon in Aquileia Mosaics - © photo by Marcella Giulia Lorenzi*

It is worth remembering here that (as it was correctly pointed out to us by the referee) one should check which are the other possible grids that – within the range of "small numbers" – best fit a cross-shape into the octagon. Figure 4 (due to the courtesy of the referee) shows that, in the range from 2 to 13, only 7 and 10 serve as good numbers of subdivisions; from the Figure we see also that 7x7 is the best choice and that it is the only one that produces the numbers 8 and 33 in a direct way (the 10x10 one generating however the square of 8). Moreover, being 7 an odd integer the 7x7 grid does have a central square around which the whole shape is rotationally symmetric, while this is no longer true for the 10x10 grid (10 being even). What really matters is how a *n*x*n* covering of the octagon by square tesserae "best fits" the images, i.e. what is the error distance from one of the octagon's top two vertices from a division line in the *n*x*n* array. Referring again to Fig. 4 we see that the smallest error is about 1% for the 7x7 grid and about 2% for the 10x10 grid. Nevertheless we believe that it is important to discuss in some details the rational approximations that produce the best fit of the 7x7 grid used by the mosaic masters.

One can also remark that $l_{square} = 1 + \sqrt{2}$ ($\approx 2.4142$) $\approx$ 24/10 $\approx$ 72/30 $\approx$ 7/3, which further explains why one can divide the side of the square (i.e., the silver number) into 7 parts that approximately split into two "lateral" and three in the center (i.e., the unit side of the octagon). Because of these numbers, the "big" enveloping square will be divided into 7 x 7 = 49 smaller (alternate black and white) squares, of which exactly 33 will reside in the central Cross and the remaining 16 will fill the four squares cut at the four vertices of the enveloping square. Out of these 16 small squares, we see that 8 are outside and 8 inside, in turn divided into two groups of 4 + 4 whole squares and 8 + 8 (triangular) halves across the edges. One might also offer a slightly different viewpoint, by remarking a further rational approximation for the silver number: i.e., $l_{square} = 1 + \sqrt{2}$ ($\approx$ 2.4142) $\approx$ 24/10 $\approx$ 12/5 = 2 + 2/5, while $\sqrt{2}$ can also be approximated by $\sqrt{2}$ ($\approx$ 1.4142) $\approx$ 14/10 = 2 x 7/10. As a consequence, we see that using the different approximation of the silver number as $1 + \sqrt{2} \approx 7/3 = 2 + 1/3$ shows that the error done is less than 7 % of the exact measure, being just 2/5 – 1/3 = 1/15.

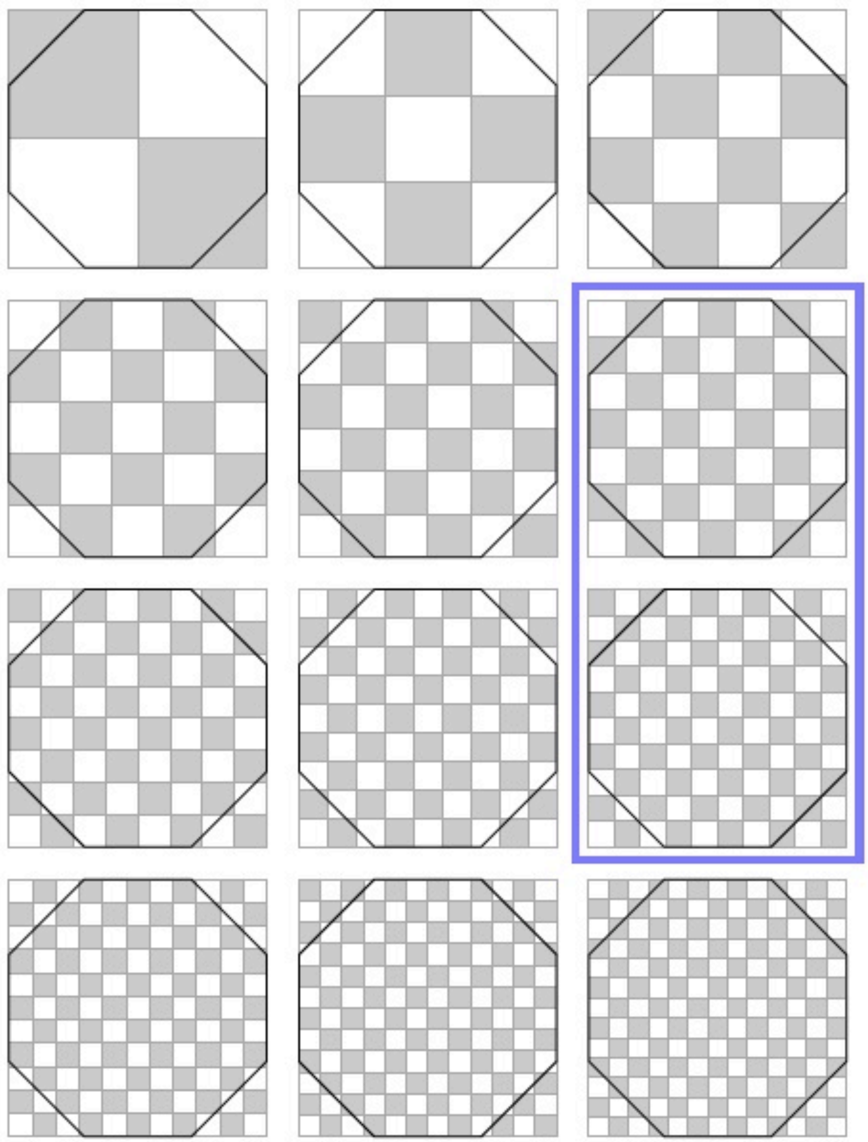

**Figure 4**: *Regular octagon superimposed on nxn checkerboards (2-13) – the two octagons with vertices closest to the lattice (i.e., 7 and 10) are highlighted*

Instead of working on linear sizes, one can also argue about the superficial sizes. In this case one sees immediately that the four triangles that are cut away have cathetus $1/\sqrt{2} = \sqrt{2}/2$ ($\approx$ 0.7071) and hypotenuse equal to one, so that they cover a unit area (since each one of them covers an area equal to ¼). Since the enclosing square has total area $A_{square}$ equal to the square of the silver number, i.e. $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$ ($\approx$ 5.8284), the area $A_{octagon}$ of the octagon is given by $A_{octagon} = A_{square} – 1 = (3 + 2\sqrt{2}) - 1 = 2 + 2\sqrt{2} = 2(1 + \sqrt{2})$ ($\approx$ 4.8284), as is well known from Euclidean geometry. On the other hand, dividing the octagon into 9 parts as in Figure 2, we have four triangles with area $A_{triangle}$ equal to 1/4, four rectangles with area $A_{rectangle}$ equal to $1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) and the central "small" square of side 1 and unit area as well. The rectangles and the small square form together the central Cross, whose total area $A_{cross}$ is therefore given by $A_{cross} = A_{octagon} - 1 = A_{square} - 2 = 1 + 2\sqrt{2}$ ($\approx 3.8284$).

We can now investigate the properties and some rational approximations of the ratios between these areas. We have first an important relation: $A_{square}/A_{octagon} = A_{square}/(A_{square} - 1) = 1 + 1/(A_{square} - 1) = (3 + 2\sqrt{2}) / (2 + 2\sqrt{2}) = - (3 + 2\sqrt{2})( 2 - 2\sqrt{2}) / 4 = 1/2\ (1 + \sqrt{2}) \approx 1.2071 \approx 6/5 = 1 + 1/5$, that shows that the remainder is $1/(A_{square} - 1) \approx 1/5$ (or, in other words, that the ratio $A_{square}/A_{octagon}$ exceeds 1 of about 20%). One has another important relation $A_{square}/A_{cross} = A_{square}/(A_{square} - 2) = 1 + 2/(A_{square} - 2) = (3 + 2\sqrt{2}) / (1 + 2\sqrt{2}) = - (3 + 2\sqrt{2})(1 - 2\sqrt{2}) / 7 = 1/7\ (5 + \sqrt{2}) \approx 1.5224 \approx 6/5 \approx 3/2 = 1 + 1/2$, from which we see that the remained is $2/(A_{square} - 2) \approx 0.5224 \approx 1/2$ (or, in other words, that the ratio $A_{square}/A_{cross}$ exceeds 1 of approximately 1/2, so that the area of the enveloping square is approximately 50% larger than the area of the central Cross, since their ratio is – again - approximately 3/2).

These are the rational approximations that can be read into the 7x7 grid that the mosaic masters were faced with. Proportions to which one is naturally led if – as we like to believe – one wants to cope with the two fundamental sacred numbers that spring up in a "natural" way: 33 (the age of Christ, and even more strikingly, in the Cross…!) and 49 (the 7th "square number", where 7 is the "number of wisdom" [1] and, in Christianity, the number of the Holy Spirit).

We also remark that one has in fact to work with square tessellations, so that only quadratic numbers can be used. Dividing the "big" enveloping square in smaller squares totals, in fact, exactly 49 "tesserae". On the other hand one has $A_{square}/A_{octagon} \approx 6/5 = 48/40$; and 48 is not a "square number". Of course, the nearest square number is 49, so that one can approximate 48/40 by 49/41 according to: $49/41 \approx 1.1951 \approx 1.2 \approx 6/5 = 48/40$. This explains why 41 among the 49 small squares eventually fill the octagon at a rather close approximation. More precisely, among the 49 squares that form a tessellation into the enclosing square one will use 41 inside the octagon (33 full squares + 16 halves) while 8 more (4 + 8 halves) will instead remain outside the octagon. The arithmetical reason why the number 33 springs up is now very simple: the square is approximately 50% larger than the Cross, so that the overall number of tesserae that covers the enclosing square (i.e., 49) splits naturally as 33 + 16, being the integer 16 (i.e., 32/2) about half of the integer 33; to state this in other words, one has $A_{square}/A_{cross} \approx 3/2 \approx 49/33$ ($\approx 1.4848$) with an error of $3/2 - 49/33 = (99 - 98)/66 = 1/66$ (i.e., less than 1 %). According to these simple rational approximations, 33 small squares will eventually fill up the central Cross (according to the decomposition 6 + 6 + 6 + 6 + 9 = 4 x 6 + 9 in the five pieces that do form the central Cross, as we mentioned above).

Reverting now to "exact proportions" of the mosaic we are analyzing, we also understand that the mosaic makers had in fact to cope with slightly different sizes (to be calculated in a slightly different way). In fact, splitting the Cross by three columns and three rows would require to use the rational number 1/3 as measure, i.e. the periodic number $0,\underline{3}$. On the other hand, dividing the left and right portions of the octagon into two columns, would require to use $1/2 \times 1/\sqrt{2} = \sqrt{2}/4$ ($\approx 0.3535$) as fundamental measure. These two lengths are of course incommensurable, since one is a rational number (1/3) and the other involves the square root of 2. The relative difference between these two units is fortunately very small: it is in fact $\sqrt{2}/4 - 0,\underline{3} \approx 0.3535 - 0,3333 \approx 0.0202$, i.e. we have about 2 % of difference; something that is difficult to see when looking at the mosaic from a distance, while using bigger sizes for the figures (or finer subdivisions, based on 4/6 instead of 2/3, or even finer ones) would likely result in a less harmonious decomposition and in bigger discrepancies on units.

Let us also remark that a fully analogous decomposition involving the same fundamental numbers (8, 33, 41 and 49) is in fact visible in an adjacent mosaic to the left of this octagon, in which curvilinear triangles were used instead of small squares although the same overall structure and proportions are fully respected (see Fig. 5).

## Conclusion: 33 and 41 from the Octagonal Mosaic

We are now ready to conclude with a further travel into the "sacred meanings" of all these numbers. These simple and extremely "natural" rational approximations (that, we stress once more, are strictly related with the rational approximations of the silver number and of the "square root of 2") led the mosaic masters to split the octagon (which by itself has already a profound esoteric meaning) to generates a Cross with a tessellation of 33 small squares – 17 of them being white and 16 black, in which we like to consider the small prevalence of white ones against the black ones as a signal of the prevalence of good against evil). The overall total 41 of small black and white Squares (some of them being suitably split in triangular halves along the oblique edges of the octagon itself) is therefore seen as the sum 41 = 33 + 8, a decomposition in which the tessellation of the octagon is formed by the sum of two rather important numbers of Christianity: the number 8 and the number 33.

As a purely numerical and Pythagorean coincidence we remark that 41 is the 13$^{th}$ Prime Number and is also related to the octagon as the 4$^{th}$ *"octahedral gnomon"*, i.e., the growing figure of so-called *"octaedral numbers"* - [1].

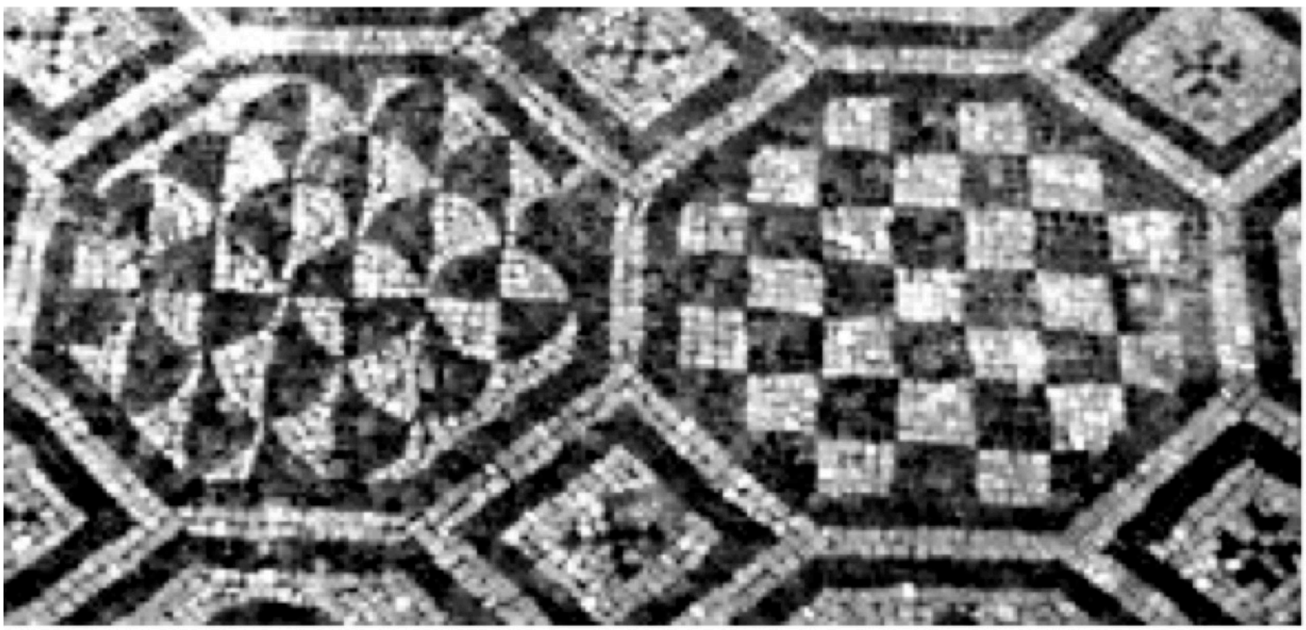

**Figure 5**: *Curvilinear Tessellation of the Octagon in Aquileia - © photo by Marcella Giulia Lorenzi*

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