Two-color Fractal Tilings

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Abstract

A variety of two-color fractal tilings (*f*-tilings) are described, in which no two adjacent tiles have the same color. Two-colorable examples from *f*-tilings that have been described previously are identified, and two techniques are used for converting *f*-tilings that are not two-colorable into new *f*-tilings that can be so colored. In the first of these, tiles are combined in order to change the valence of vertices to all be even, ensuring two-colorability. This technique is applicable to a limited number of *f*-tilings and can result in prototiles with an infinite number of edges and corners. In the second technique, tiles are divided into two or more smaller tiles such that all vertices of the new *f*-tiling have even valence.

1. Introduction

Fractals and tilings can be combined to form a variety of visually appealing constructs that possess fractal character and also obey many of the properties of tilings. Previously, we described families of fractal tilings based on kite- and dart-shaped quadrilateral prototiles [1], v-shaped prototiles [2], prototiles that are segments of regular polygons [3], prototiles derived by dissecting polyhexes [4], prototiles derived by dissecting polyominoes [5], and prototiles derived by dissecting polyiamonds [6]. A prototile is a tile to which many or all of the tiles in a tiling are similar; i.e., the same within scalings, rotations, and reflections. Many of these fractal tilings may be viewed online [7]. These papers appear to be the first attempts at a systematic treatment of this topic, though isolated examples were earlier demonstrated by M.C. Escher [8] and Peter Raedschelders [9]. Chung and Ma have more recently described the decoration of fractal tilings using invariant mappings [10].

In Grünbaum and Shephard's book *Tilings and Patterns* [11], a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. The constructs discussed in this paper do not for the most part cover the entire Euclidean plane; however, they do obey the restrictions on gaps and overlaps. To avoid confusion with the standard definition of a tiling, these constructs are referred to as "*f*-tilings", for fractal tilings. These *f*-tilings contain an infinite number of tiles but are finite in extent.

The tiles used here are "well behaved" by the criterion of Grünbaum and Sheppard; namely, each tile is a (closed) topological disk. Most of the *f*-tilings explored in [1]–[6] and all of those described in this paper are edge-to-edge; i.e., the corners and edges of the tiles coincide with the vertices and edges of the tilings. However, they are not "well behaved" by the criteria of normal tilings; namely, they contain singular points, defined as follows. Every circular disk, however small, centered at a singular point meets an infinite number of tiles; i.e., the tiles become infinitesimally small as such points are approached. These *f*-tilings provide a rich source of unique fractal images and also possess considerable recreational mathematics content.

The subject of this paper is the coloring of *f*-tilings with two colors ("two-coloring") such that no adjacent tiles have the same color. An *f*-tiling that allows such a coloring will be referred to as two-colorable. Black and white are the two colors used in the figures. Esthetically, there are several reasons for exploring two-color *f*-tilings. Black-and-white designs are bold, with a strong graphic sense. The absence of color removes a source of distraction and allows a pure focus on form. Black-and-white tilings also allow ground/figure reversal which is effective in optical illusions. Escher employed this feature in some of his best-known prints, including "Day and Night" (1938), "Sky and Water I" (1938), and "Metamorphosis III (1967-1968) [8]. Black and white, being opposites with associations such as night and day, evil and good, and ignorance and knowledge, are also well-suited to portraying dualities, as done by Escher in "Circle Limit IV" (1960, also known as "Heaven and Hell").

In the following sections, a survey is made of *f*-tilings to determine which ones allow two-coloring. We then describe how to modify *f*-tilings that do not allow two-coloring so that the modified versions do allow two-coloring.

2. Earlier Fractal Tilings that Allow Two-Coloring

In order for a tiling to be two-colorable, every vertex must have even valence. I.e., an even number of tiles must meet at each vertex, since the coloring of those tiles must alternate between black and white. Any tiling that meets this condition is two-colorable [12].

Examining the various *f*-tilings described in [1]–[6] and others [13] reveals that most *f*-tilings do not meet this condition on their vertices. All *f*-tilings with *n*-fold rotational symmetry with odd *n* fail, since *n* tiles meet in the center. Many of the *f*-tilings with even *n* fail due to an odd number of tiles meeting at some of the other vertices. *f*-tilings with more than six edges won't be considered here, since there are a large number of these and most are relatively uninteresting variations on prototiles with fewer edges. In addition, only *f*-tilings possessing a single prototile are considered here.

Of all the *f*-tilings described in [1]–[6] having a single prototile with six or fewer edges, the only ones found in the course of this work to be two-colorable either have 8- or 12-fold rotational symmetry and kite- or dart-shaped prototiles, or they have V-shaped prototiles. The kite and dart examples with 8-fold symmetry are shown in Fig. 1. In this and subsequent figures only a portion of the full *f*-tilings are shown to allow smaller generations of tiles to be more visible. The lower right corner of the groups of tiles shown in Figs. 1 and 2 would be the centers of the full *f*-tilings.

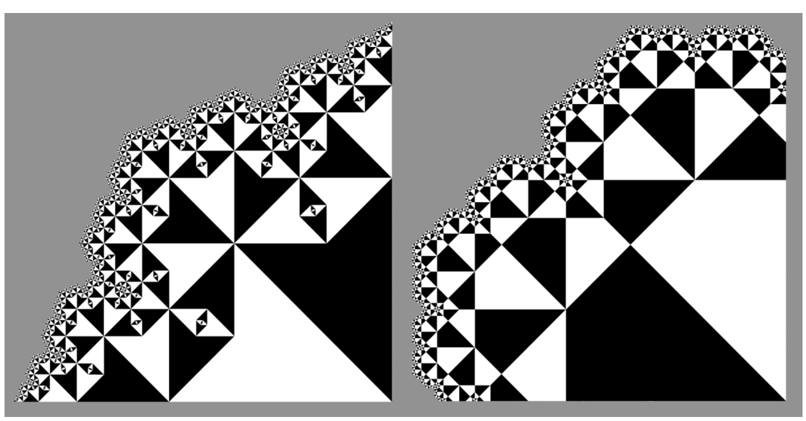

Figure 1: *Two-color f-tilings with 8-fold rotational symmetry based on kite (left) and dart (right) prototiles.*

The two-colorable *f*-tilings with V-shaped prototiles are of two types. There are two *f*-tilings of the first type, as shown in Fig. 2. There is an infinite family of *f*-tilings of the second type, though they are relatively uninteresting. The first two members of this family are shown in Fig. 3, from which it is clear that the 8-, 10-, 12-, … fold versions will also be two-colorable. Note that the rotational symmetry given for these *f*-tilings does not take coloring into account. E.g., an 8-fold *f*-tiling would become 4-fold if the two-coloring were required to be invariant under rotation.

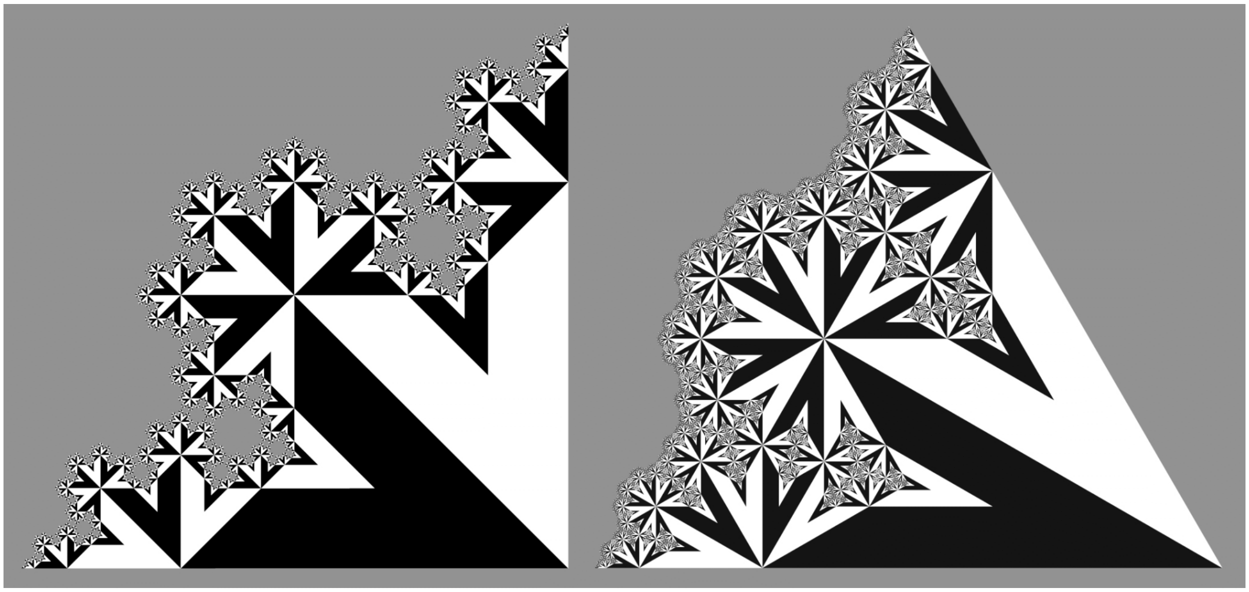

Figure 2: *Two-color f-tilings of the same general type, with 8-fold and 12-fold rotational symmetry, respectively.*

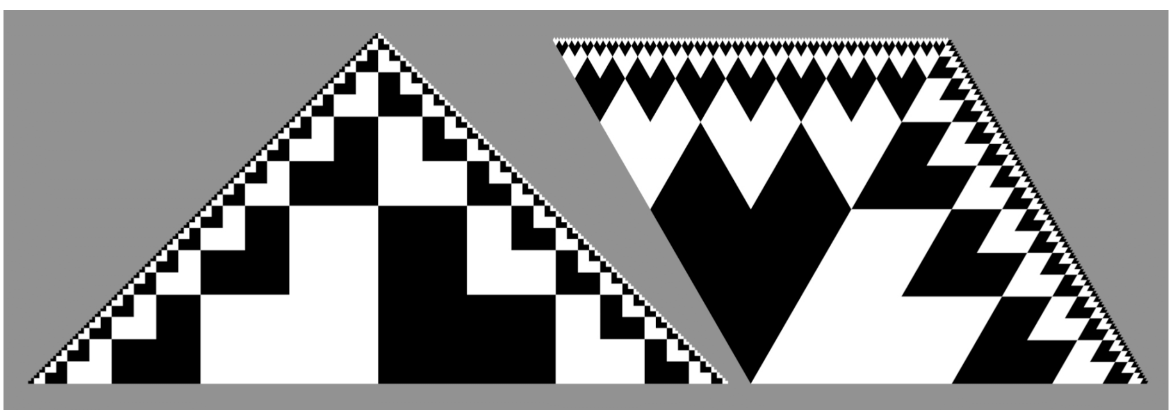

Figure 3: *The first two members of an infinite family of two-color f-tilings of the same general type, with 4-fold and 6-fold rotational symmetry, respectively.*

3. Combining Tiles in Earlier *f*-tilings to Allow Two-Coloring

Since there are so few single-prototile *f*-tilings that are two-colorable, it is natural to ask if it might be possible to modify existing *f*-tilings to create new two-colorable *f*-tilings. This is equivalent to asking if they might be modified such that every vertex takes on an even valency, in a manner that retains the edge-to-edge and single-prototile character. One strategy is to combine tiles. In the course of this work, this strategy was generally not found to be successful, but there are cases for which it works. A 6-fold rotationally symmetric *f*-tiling is shown in Fig. 4, with two different new prototiles formed by combining dart-shaped tiles. Edge-to-edge *f*-tilings can be formed from both protiles, as shown in Figs. 5a and 5b. In Fig. 5b, the tiles are mirrored between successive generations. Note that these prototiles, in contrast to those described above, contain an infinite number of corners and edges. The holes in the *f*-tilings of Figs. 4, 5a, 5b, and 7 all fill in to singular points in the limit.

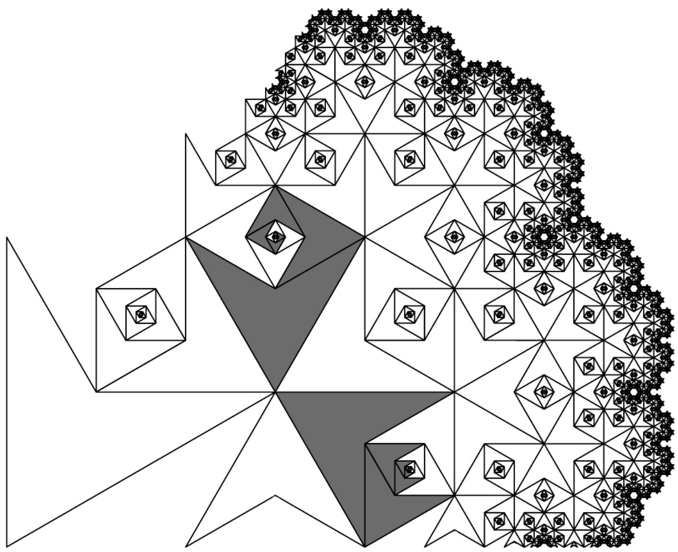

Figure 4: *Two prototiles (shaded gray) formed by combining dart-shaped tiles in an f-tiling with 6-fold rotational symmetry (1/6 of the full f-tiling is shown here).*

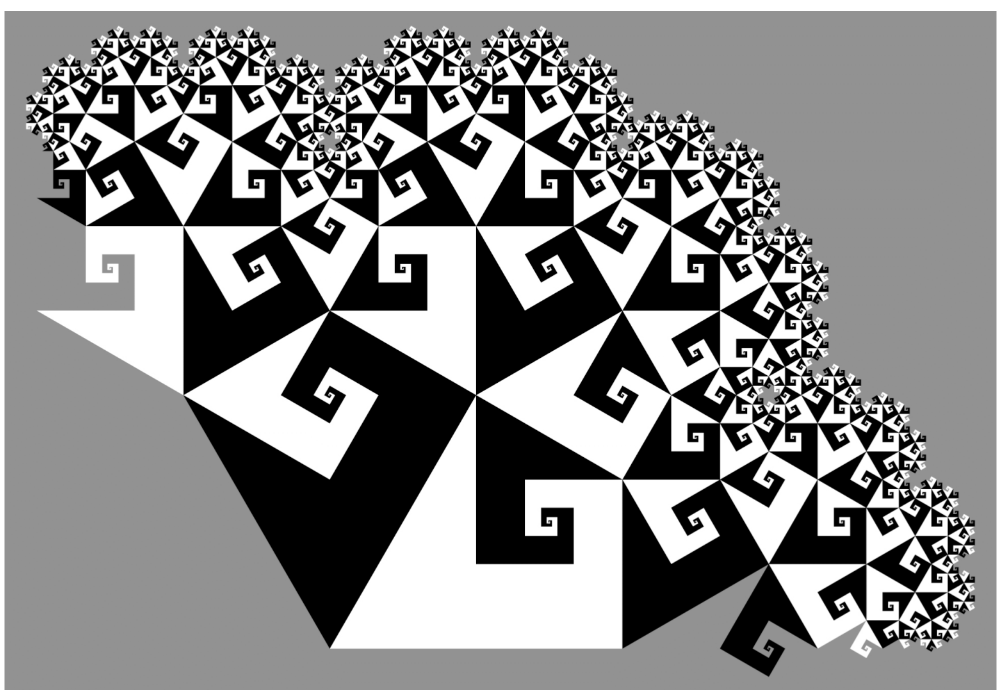

Figure 5a: *A two-color f-tiling formed from a 6-fold f-tiling of darts by combining tiles as indicated by the upper gray-shaded region in Fig. 4.*

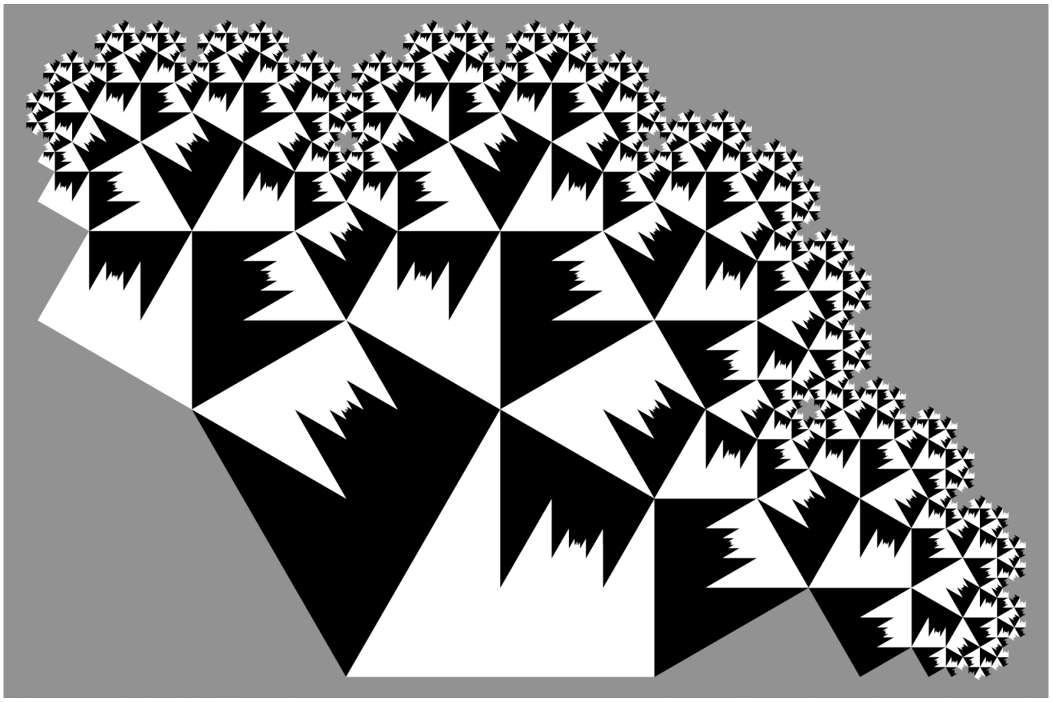

Figure 5b: *A two-color f-tiling formed from a 6-fold f-tiling of darts by combining tiles as indicated by the lower gray-shaded region in Fig. 4.*

Another example is shown in Fig. 6, where the original *f*-tiling, which has four-fold rotational symmetry, is based on a tile created by dividing an octomino into four congruent parts.

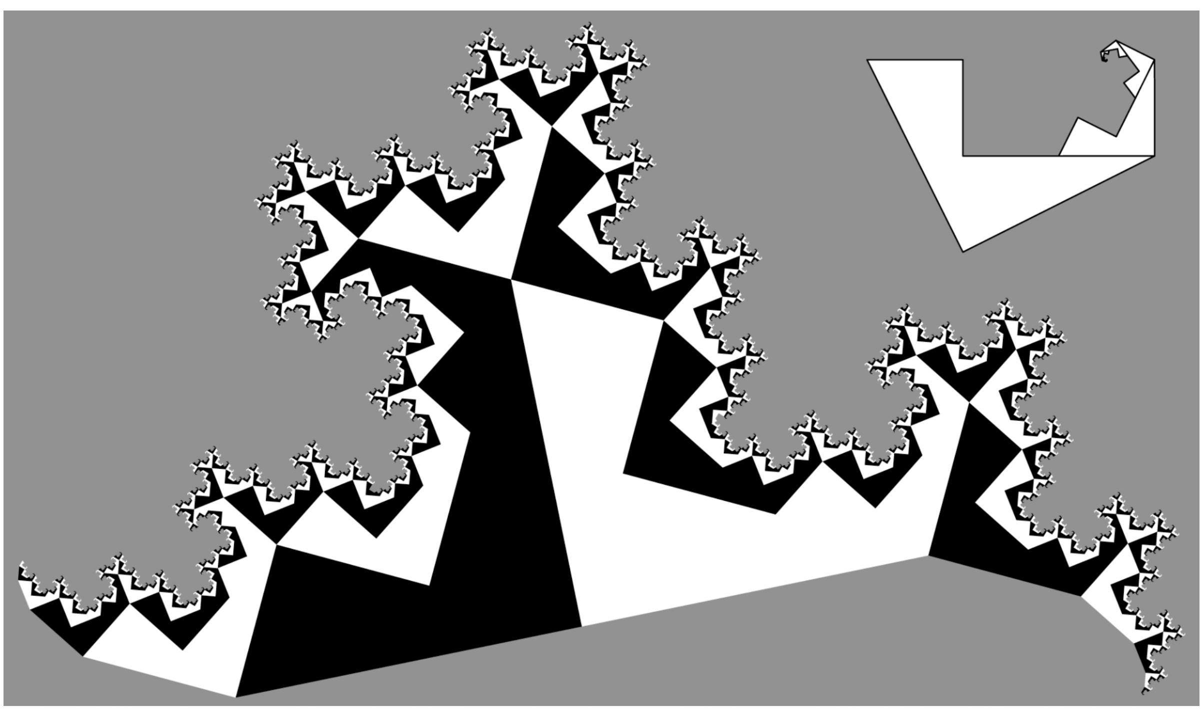

Figure 6: *Two-color f-tiling formed by combining tiles from an earlier f-tiling, as shown in the inset.*

4. Splitting Tiles in Earlier *f*-tilings to Allow Two-Coloring

A second strategy is to split the original prototile into two or more smaller prototiles such that all of the new prototiles are similar to one another. Again, this is not possible for most of the *f*-tilings examined, but it is possible in a few cases. One of these is the dart-shaped prototile shown in Fig. 4, which can be divided along the axis of mirror symmetry to form a triangular prototile. Analogous *f*-tilings with dart-shaped prototiles with 8-fold and 12-fold rotational symmetry also work. Another is an *f*-tiling with a kite-shaped prototile and 6-fold rotational symmetry, where the kite can be divided along the axis of mirror symmetry. The original *f*-tiling, the new one with split tiles, and an Escheresque version of the split-tile *f*-tiling are all shown in Fig. 7. Note that the black-and-white triangles are mirrored from one generation to the next. Analogous *f*-tilings with kite-shaped prototiles and 8-fold and 12-fold rotational symmetry also work. These kite- and dart-based *f*-tilings are described in detail in [1].

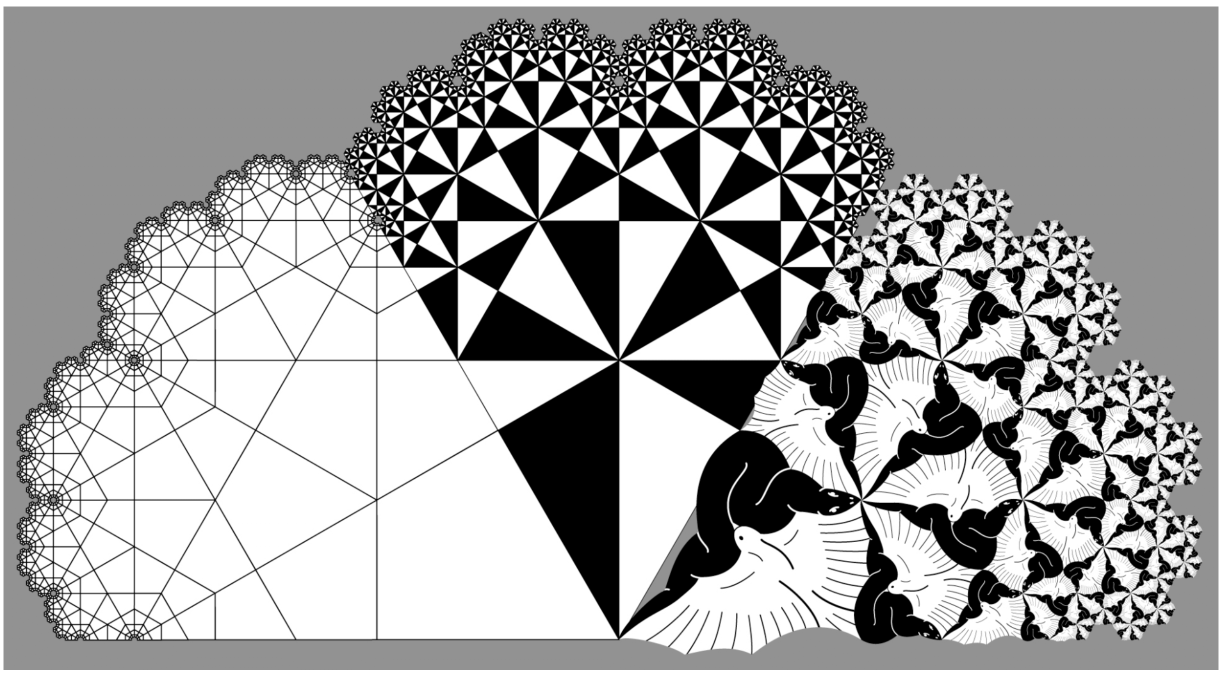

Figure 7: *A two-color f-tiling (center portion) with six-fold symmetry formed by dividing kite-shaped tiles from an earlier f-tiling (left portion). The right portion shows an Eschereque f-tiling of birds and snakes based on the two-color f-tiling at center.*

If two or more prototiles are allowed, then splitting of tiles can be applied to a much wider range of *f*-tilings. This topic hasn't been systematically explored, but two examples that demonstrate some of the possibilities are shown in Fig. 8, where the original *f*-tiling was based on a V-shaped prototile. In the first, straight lines are drawn between selected corners of the original prototile to divide it into two similar triangular tiles and a dart-shaped tile. In the second, a portion of a circle is use to connect two corners of the original prototile, creating two tiles. The use of curves creates more visual interest in this case.

Another example is shown in Fig. 9, where the starting *f*-tiling was created by dividing an octomino into four congruent parts. The original prototile was divided into two pairs of similar triangles to create the two-color *f*-tiling shown in the lower portion of the figure. Using curves instead of straight lines to divide the original prototile results in the two-color *f*-tiling shown in the top portion of the figure. Note how the curves change the graphic character of the design from a starkly geometric one to a stylized organic one.

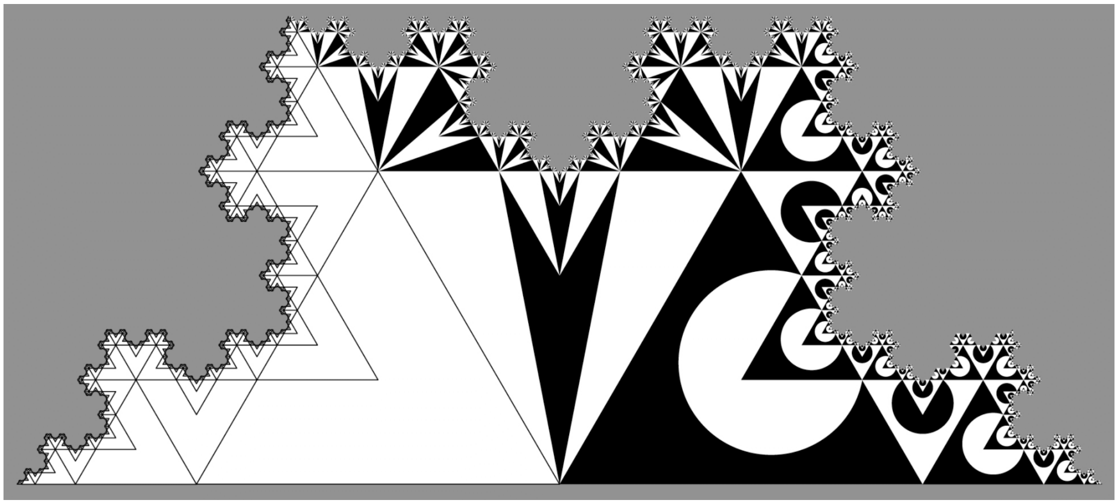

Figure 8: *A two-color f-tiling (center portion) with six-fold symmetry formed by dividing V-shaped tiles from an earlier f-tiling (left portion) into two similar triangles and a dart. The right portion shows an alternative division of the V-shaped tile into two new prototiles.*

Any edge-to-edge tiling with a prototile having an even number of corners can be turned into a two-color tiling by placing a point in the interior of the tile and drawing lines from that point to each of the corners. This has the effect of doubling the number of tiles meeting at each of the original vertices, ensuring they all have even valence. In addition, since the tile has an even number of corners, the number of lines radiating from the interior point is also even, so that the new vertices also have even valence. This is illustrated in Figure 10, where a trapezoidal prototile is divided by introducing the point P.

6. Conclusions

A variety of two-color *f*-tilings have been described here. While most *f*-tilings that have been described previously are not two-colorable, two techniques have been used for converting them into two-colorable *f*-tilings. In the first of these, tiles are combined in order to change the valence of vertices to all be even. This technique is applicable to a limited number of *f*-tilings and can result in prototiles with an infinite number of edges and corners. In the second technique, tiles are divided into two or more smaller tiles such that all vertices of the new *f*-tiling have even valence. There is a considerable degree of freedom in how this is done, allowing different graphic effects to be realized. Curved lines can be introduced to soften designs, and Escheresque versions with lifelike motifs are also possible. One direction for further exploration would be to examine three-colorable *f*-tilings.

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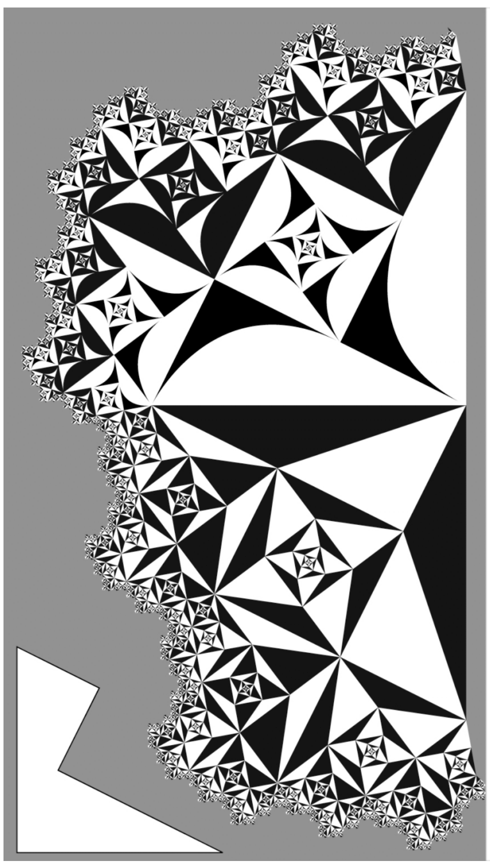

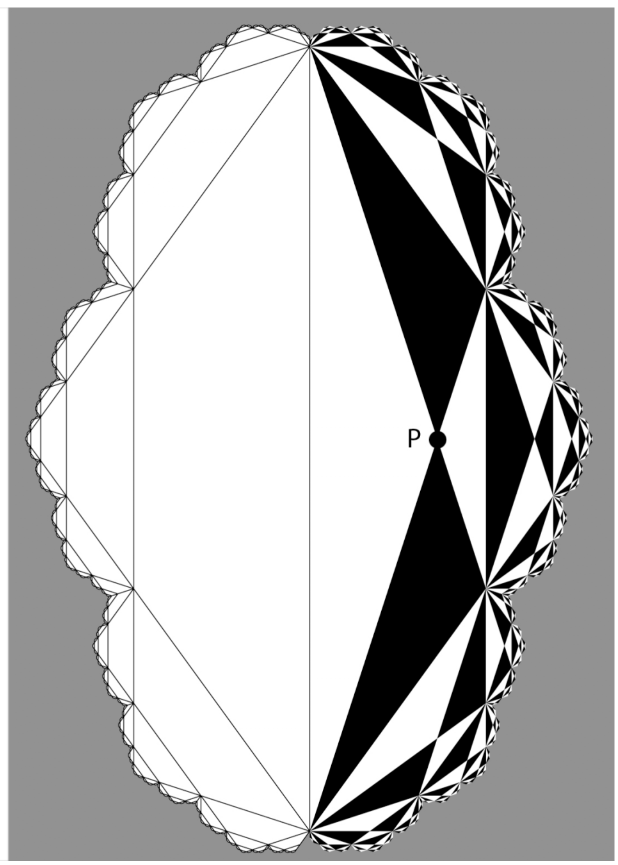

Figure 9: *Portions of two-color f-tilings formed by dividing the prototile shown at lower left from an earlier f-tiling using straight lines into two pairs of similar triangles (lower portion) and using curved lines (upper portion).*

Figure 10: *A two-color f-tiling formed by dividing the trapezoidal prototile from an earlier f-tiling (left half) into two pairs of similar triangles. A new vertex is introduced at point P and similar points in each tile in the original f-tiling.*