Sinuous Meander Patterns in Natural Coordinates

David Chappell
Math/Physics/Computer Science Department
University of La Verne
1950 3rd Street
La Verne, CA, 91750, USA
E-mail: dchappell@laverne.edu

Abstract

Natural (or intrinsic) coordinate systems parameterize curves based on their inherent properties such as arc length and tangential angle, independent of external reference frames. They provide a convenient means of representing many organic, flowing curves such as the meandering of streams and ocean currents. However, even simple functions written in natural coordinates can produce surprisingly complex spatial patterns that are difficult to predict from the original generating functions. This paper explores multi-frequency, sine-generated patterns in which the tangential angle of the curve is related to the curve's arc length through a series of sine functions. The resulting designs exhibit repeating forms that can vary in subtle or dramatic ways along the curve depending on the choice of parameter values. The richness of the "pattern space" of this equation suggests that it and other simple natural equations might provide fertile ground for generating geometric, organic and even whimsical patterns.

Introduction

Consider the path of the Alaskan stream shown in Figure 1. It invites you to trace over it with your eyes, following the folds and curves. You scan the repeating bends looking for patterns and variations. The forms repeat just enough to suggest that there might be some hidden, deeper order to be discovered if you just look a little longer. The curve has hints of symmetry, yet not enough to render it lifeless. While symmetry can be a powerful organizing principle in visual art, it can also be stifling, leaving little room for individual character. Carol Bier [1] refers to the "tyranny of the repeat," a term coined by fiber artist, Katherine Westphal [2]. As Bier suggests, beauty arises through variation and symmetry breaking more often than it does through symmetry alone. In this paper I generalize a simple equation historically used to model river meanders in order explore the tension between rigid mathematical symmetry and the "playfulness" of organic variation.

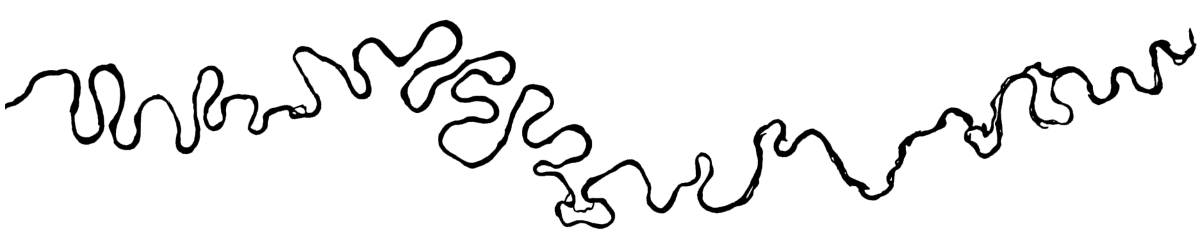

Figure 1: *Sketch of the path of a meandering stream near Fort Yukon, Alaska derived from [3].*

Natural coordinate systems parameterize curves in terms of variables inherent to the curve itself. Planar curves parameterized by the tangential angle θ and arc length s are known as Whewell equations [4]. Once the $\theta(s)$ function is specified, the Cartesian (x, y) coordinates may be obtained by integration:

$$x(s) = \int_0^s \cos\theta(s')\,ds'$$

$$y(s) = \int_0^s \sin\theta(s')\,ds'$$

This paper will explore a Whewell equation in which the tangential angle θ is written as the sum of N sine functions:

$$\theta(s) = \sum_{i=1}^{N} A_i \sin\left(\frac{2\pi}{\lambda_i} s + \varphi_i\right).$$

In general, each sine term has three parameters: the amplitude A_i, the wavelength λ_i and the phase constant φ_i. The phase constant is set to zero in this paper to reduce the number of free parameters. Because only wavelength ratios affect the form of the pattern and since we are not concerned with the overall scale of the curve, the number of free wavelength parameters is $N-1$. Thus, the total number of free parameters including the amplitudes is $2N-1$. The system of equations was solved by numerical integration using the midpoint rule and Simpson's rule. The methods produced consistent and stable results over the range of step sizes employed. In the following sections, patterns will be examined containing one, two and three sine terms.

Single-Frequency Patterns

In 1849, Whewell [4] studied a family of intrinsic curves generated by a single sine function:

$$\theta(s) = A\sin\left(\frac{2\pi}{\lambda} s\right).$$

These curves were later called *sine-generated* curves by Leopold [5]. Sine-generated curves have been suggested as a simple model for meandering streams [5-7], as an approximate solution to the bending of elastic rods described by Euler's *elastica* [8,9], and as a tool for generating curved shapes used in designing font outlines [10]. Sine-generated curves were explored in three dimensions by [7]. In the present paper I extend sine-generated curves to include multiple frequency components.

The wavelength λ controls the spatial scale of the curve, as may be verified through a change of variables $u=s/\lambda$ with $du=ds/\lambda$. One immediately finds that the parametric curve $(x(u), y(u))$ linearly scales with the wavelength λ. The shape of the curve is not affected by λ.

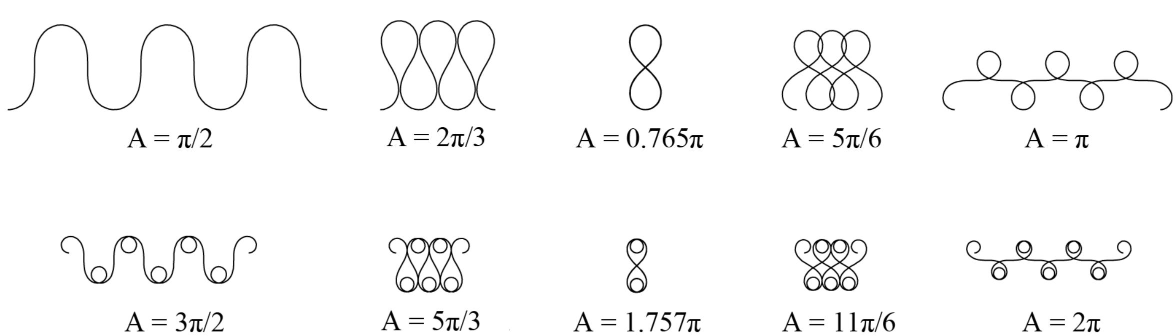

Figure 2: *Single frequency sine-generated curves as a function of the amplitude A. The curves were obtained through numerical integration.*

The amplitude A controls the shape of the curve. The maximum tangential angle the curve makes with the positive x-axis is $|\theta_{\max}| = A$. Thus, when $A = 0$, the curve reduces to a straight line along the x-axis. As the amplitude increases, the curve develops periodic oscillations. When $A = \pi/2$, the tangent line to the curve periodically becomes vertical (see Figure 2). As A approaches π, the curve develops an alternating sequence of clockwise- and counterclockwise-loops. In general, when $A = n\pi$, where n is an integer, the curve completes n rotations in each loop (e.g., see the $A = 2\pi$ case in Figure 2). Closed loops are also possible. The approximate amplitudes producing the first three closed loops were found using numerical integration to be $A \approx 0.76548\,\pi$, $1.75710\,\pi$, $2.7546\,\pi$.

Winding Number and Pattern Rendering

The winding number of a plane curve around a given point P is defined as $W = (\theta_f - \theta_i)/2\pi$, where θ_f and θ_i are the final and initial angles the curve makes with respect to the x axis as measured from P. For closed curves, the winding number equals the number of full counterclockwise turns the curve makes around the point minus the total number of clockwise turns. Open curves yield fractional winding numbers. A simple method of numerically calculating the winding number at a given point may be obtained by drawing a line from the point in an arbitrary direction. The winding number equals the number of times the line intersects counter-clockwise segments of the curve minus the number of times it intersects clockwise-oriented segments (see Figure 3(a)).

The color figures in this paper were created by calculating the winding number at each pixel in the image and then applying a color palette. Color pallets vary from figure to figure based on aesthetic considerations. The muted palettes were chosen to focus attention on the spatial form of the curves rather than on color interactions. The computation time required to produce an image depends on the total arc length of the curve, but most images were constructed in less than a minute for a resolution of 4800x1600 pixels.

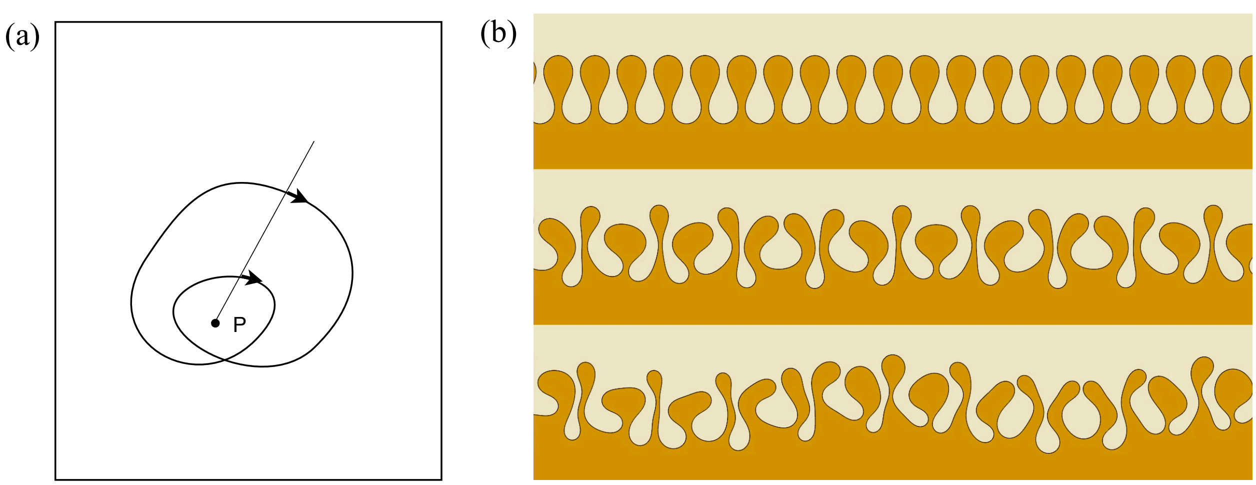

Figure 3: *(a) Numerical method of calculating the winding number at point P as described in the text. In this example the winding number is –2. (b) Non-intersecting curves colored by the winding number. Top panel: one-frequency pattern with $A_1 = 2.02$, $\lambda_1 = 1$; Middle panel: two-frequency pattern with $A_1 = 2.02$, $\lambda_1 = 1$, $A_2 = 0.47$, $\lambda_2 = 0.391$; Bottom Panel: three-frequency pattern with $A_1 = 2.02$, $\lambda_1 = 1$, $A_2 = 0.47$, $\lambda_2 = 0.391$, $A_3 = 0.266$, $\lambda_3 = 0.288$.*

Two-Frequency Patterns

Sine-generated curves with two frequency components have three independent parameters that effect the pattern shape: A_1, A_2 and the ratio λ_2/λ_1:

$$\theta(s) = A_1 \sin\left(\frac{2\pi}{\lambda_1}s\right) + A_2 \sin\left(\frac{2\pi}{\lambda_2}s\right).$$

Beats. The well-known phenomenon of beats results when the wavelengths are close: i.e., when $\lambda_2/\lambda_1 \approx 1$. The beat wavelength is given by $1/\lambda_B = |1/\lambda_1 - 1/\lambda_2|$. Figure 4b shows the resulting curves when λ_2/λ_1=1.2. The corresponding beat wavelength is $\lambda_2 = 6\ \lambda_1$. Since the beat wavelength is evenly divisible by both λ_1 and λ_2, the curves repeat exactly after each beat and the "tyranny of the repeat" reigns high.

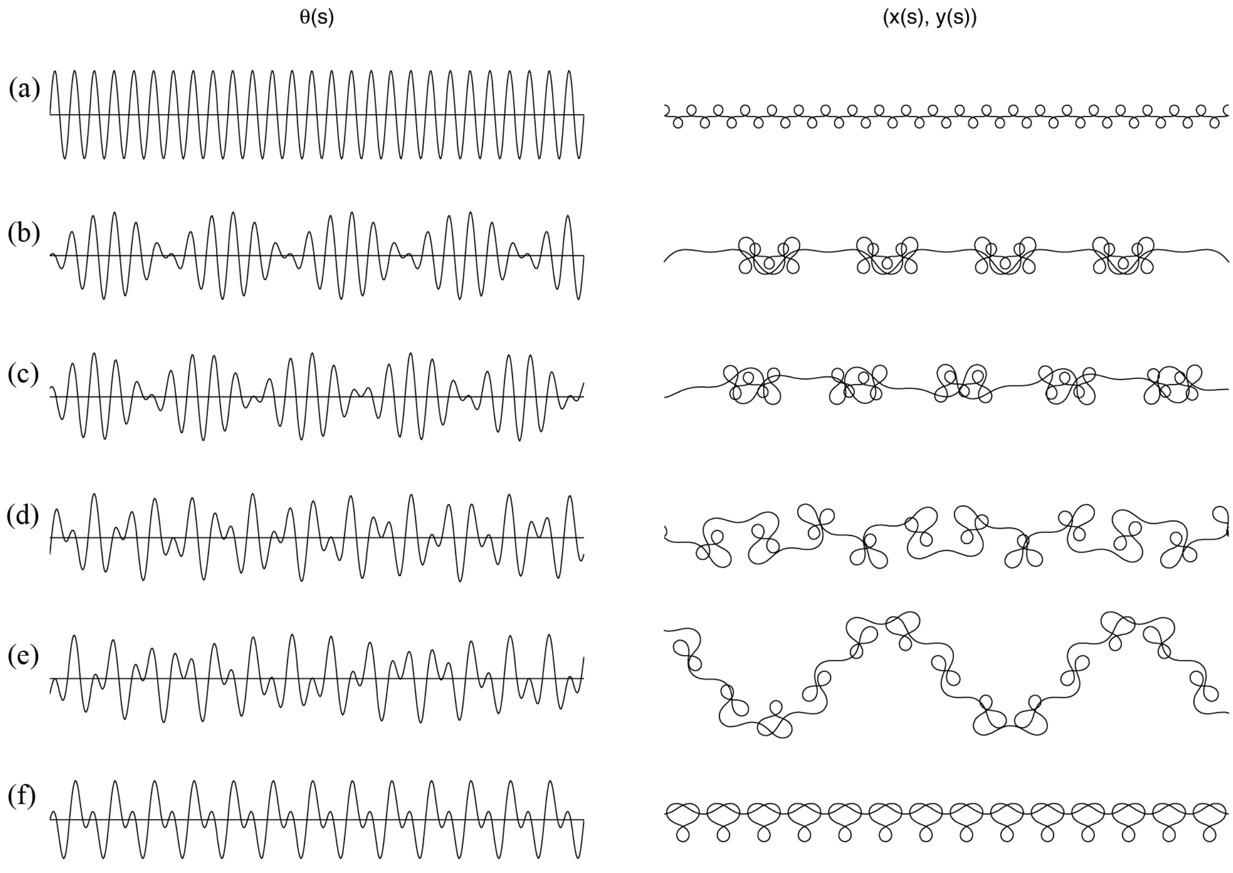

Figure 4: *Two-frequency curves displaying beats. The left column shows the θ(s) curve with the θ = 0 axis indicated. The right column shows the resulting Cartesian (x,y) pattern. The amplitude is* A_1=A_2=π/2 *for all the curves. The wavelength ratios* λ_2/λ_1 *for each pair are: (a) 1, (b) 1.2, (c) 1.23, (d) 1.61, (e) 1.85, (f) 2*

In Figure 4(c) the wavelenth ratio is given by $\lambda_2 = 1.23\ \lambda_1$. In this case, the beat wavelength is $\lambda_B \approx 5.348\ \lambda_1$, which is not divisible by λ_2. The result is a pattern that exhibits variations on a theme. The design within each beat cycle evolves until the sequence returns to the original pattern. In this example, the curve repeats exactly after 23 beat cycles.

In general, the curves will be periodic if the wavelength ratio can be written as a rational number: $\lambda_2 / \lambda_1 = n / m$, where n and m are positive integers. The ratio of the repeat cycle s_R to λ_1 is given by $s_R / \lambda_1 = n / \gcd(n,m)$, where gcd($n$,$m$) is the greatest common divisor of n and m.

As the wavelength ratio increases much above unity, the beats merge together (see Figure 4(d)). Repeating forms are clearly visible, but the variation between beats blends into a unified pattern. When the wavelength ratio approaches two, a new beat pattern begins to emerge as λ_2 beats against $2\lambda_1$. The pattern develops large amplitude undulations (see Figure 4(e)). The beat wavelength is $1/\lambda_{2B} = |2/\lambda_2 - 1/\lambda_1|$. When $\lambda_2 = 2\,\lambda_1$, the beat wavelength becomes infinite and the pattern is periodic with a wavelength of λ_2.

Symmetrical Patterns. As discussed above, the sine-generated curve is periodic if the wavelengths λ_1 and λ_2 are commensurate. Nearly periodic structures are also possible when the beat wavelength is close to an integer multiple of λ_1 and λ_2. The resulting periodic designs can exhibit a wide range of spatial forms. Two examples are shown in Figures 5 and 6. The figures are colored according to the local winding number. Visually, the dominant organizing principle of these patterns is symmetry rather than variation. Even though the sine-generated curves for these patterns contain very similar structural variations as the more organic patterns presented below, the large-scale organization of the variations in Figures 5 and 6 has a much higher level of symmetry and uniformity.

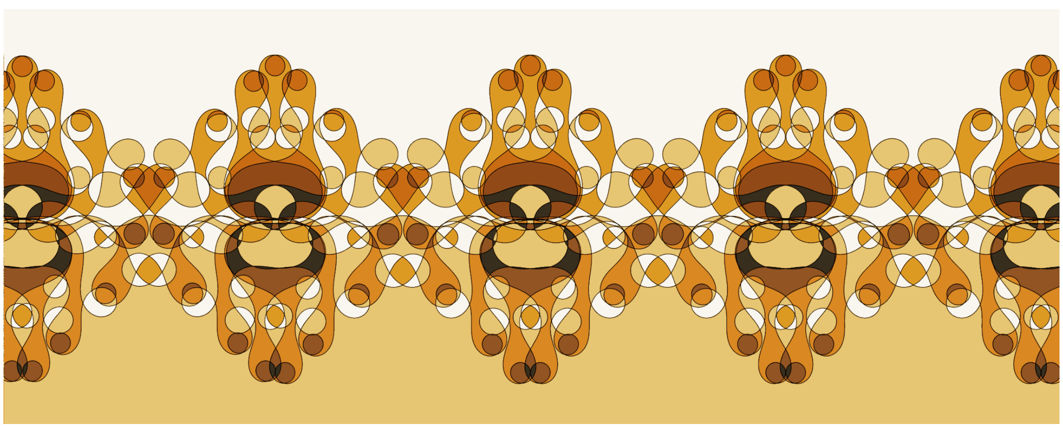

Figure 5: A *two-frequency pattern ($A_1 = A_2 = 2.76$, $\lambda_2/\lambda_1 = 1.65$)*

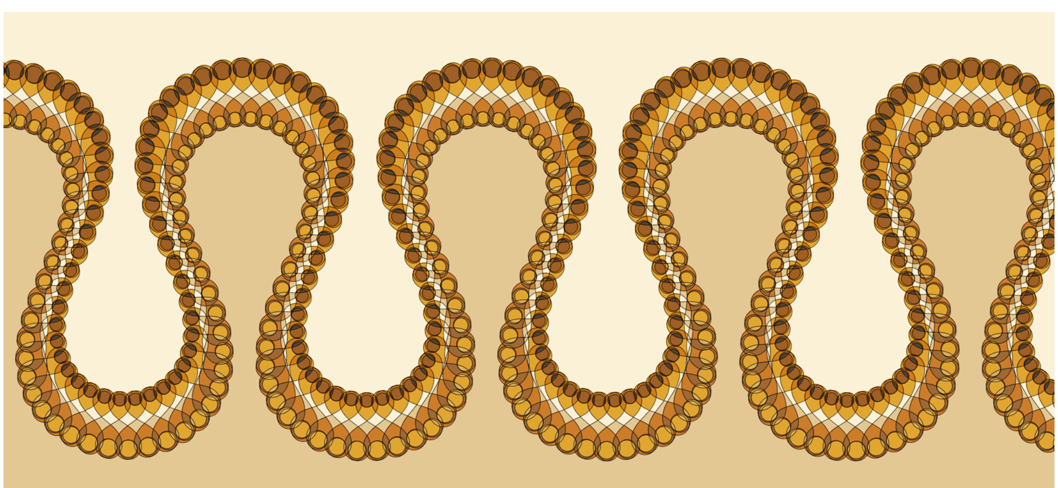

Figure 6: A *two-frequency pattern ($A_1 = 0.7754$, $A_2 = 8.71$, $\lambda_2/\lambda_1 = 1.979$)*

Variations on a Theme. Figure 7 shows a two-frequency curve that contains a number of repeating forms (which remind the author of a series of abstract rabbits). This curve is reminiscent of a series of still frames in an Eadweard Muybridge study of animal locomotion—the forms appear to evolve and move almost as if they were animated. Like many of the patterns presented in this paper, the forms vary in shape along the curve creating a succession of variations on a common theme (for other examples see Figures 8 and 9). The spatial rhythms seen in many of the figures are created by the underlying wavelengths beating against each other.

One of the striking aspects of this system is how sensitive the sine-generated patterns are to subtle changes in the *θ(s)* equation. Small changes in the parameters can lead to surprisingly diverse, qualitative changes in the visual appearance of the resulting pattern.

Figure 7: *Two-frequency pattern (A_1 = 0.7754, A_2 = 9.3736, λ_2/λ_1 =12.0881)*

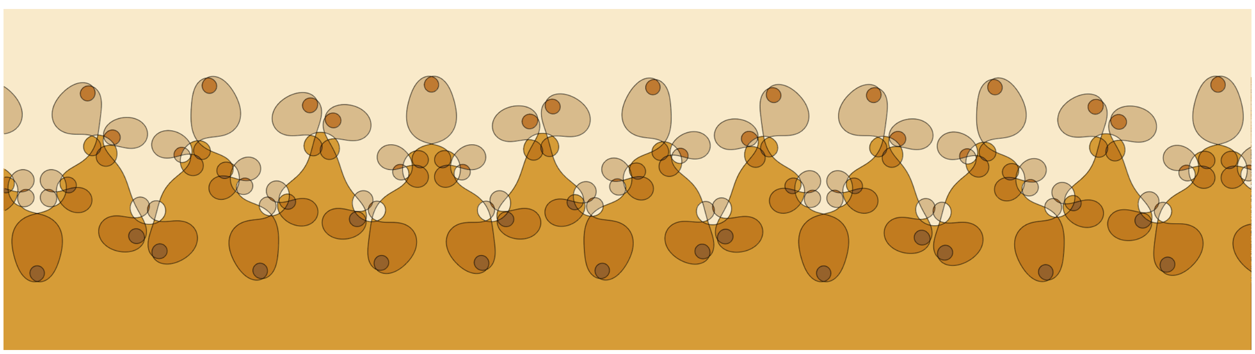

Figure 8: *Two-frequency pattern (A_1 = 0.7754, A_2 = 5.81, λ_2/λ_1 =3.72)*

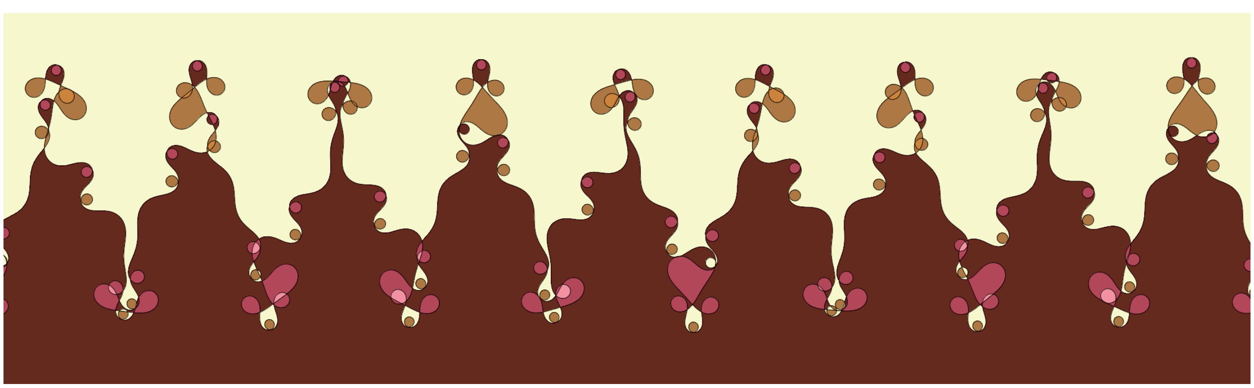

Figure 9: *Two-frequency pattern (A_1 =1.93, A_2 = 3.559, λ_2/λ_1 =1.844)*

Three-Frequency Patterns

Sine-generated curves with three frequency components have five independent parameters that affect the pattern shape: A_1, A_2, A_3 and the ratios λ_2/λ_1 and λ_3/λ_1. The generating function is

$$\theta(s) = A_1 \sin\left(\frac{2\pi}{\lambda_1}s\right) + A_2 \sin\left(\frac{2\pi}{\lambda_2}s\right) + A_3 \sin\left(\frac{2\pi}{\lambda_3}s\right).$$

As the number of frequency components increases, the sine-generated curves display a wider variety of forms. Periodic features present in a two-frequency pattern become modulated by the other frequencies, producing more complex forms (see Figures 10-13).

Consider the sequence of one-, two- and three-frequency curves shown in Figure 3b. Each of the 18 teardrop-shaped bends in the one-frequency pattern are identical. The resulting pattern has a high degree of symmetry (translational and reflection) allowing no variation between the forms. In the two-frequency pattern, the teardrops alternate between being elongated and more compact, although this alternation is also modulated by longer-period variations. The pattern almost repeats every nine teardrops, but not quite. The result is a more visually engaging image that has a kinetic quality as one's eyes move back and forth between the forms. The three-frequency pattern nearly repeats on every eleventh teardrop, but, again, not quite. More complex repetitions are present in Figures 10-13.

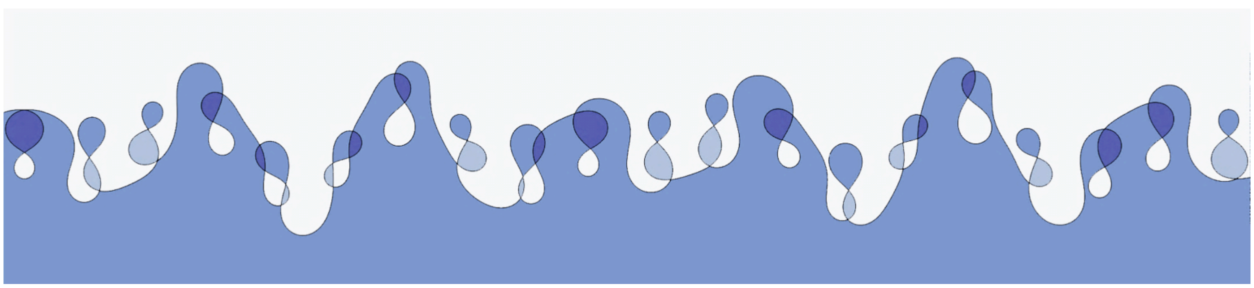

Figure 10: *Three-frequency pattern (A_1 =0.34, A_2 =1.98, A_3 =0.6, λ_2/λ_1 =3.59, λ_3/λ_1 = 1.245)*

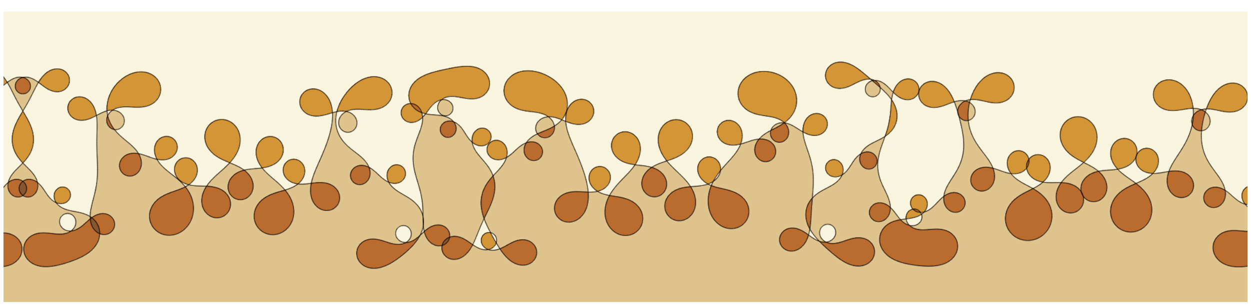

Figure 11: *Three-frequency pattern (A_1 =0.57, A_2 =2.72, A_3 =0.9, λ_2/λ_1 =1.805, λ_3/λ_1 = 1.037)*

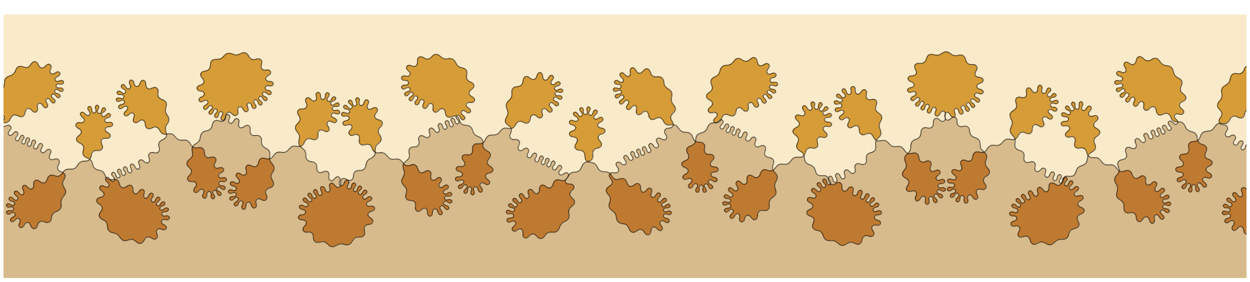

Figure 12: *Three-frequency pattern (A_1=0.691, A_2=1.1686, A_3=2.77, λ_2/λ_1=0.923, λ_3/λ_1=32.45)*

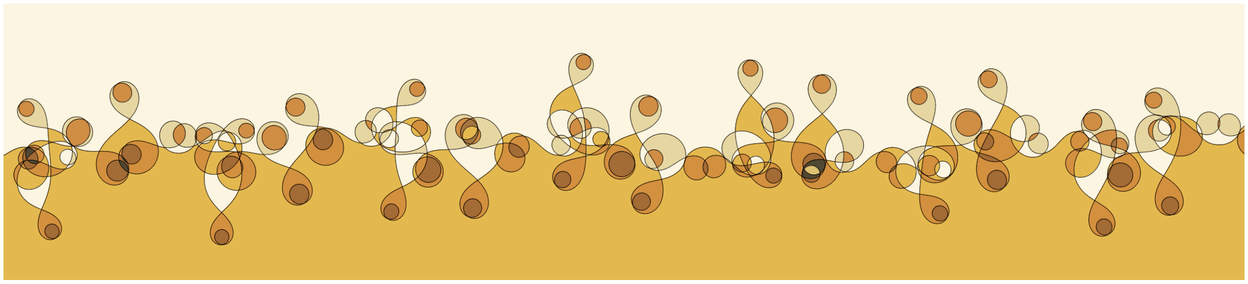

Figure 13: *Three-frequency pattern (A_1=0.434, A_2=0.812, A_3=5.866, λ_2/λ_1=1.083, λ_3/λ_1=2.798)*

Conclusion

Multi-frequency, sine-generated curves offer a rich environment for creating periodic and quasi-periodic spatial patterns. The patterns range from symmetrical forms to repeated shapes that evolve through a sequence of variations on a common theme. The variations invite comparison, feeding our innate tendency to look for common traits and structures. Some of the curves allude to figurative or "Dr. Seussian" shapes, introducing an element of humor and playfulness into an otherwise abstract investigation. While the curves under study were inspired by equations historically used to model river meanders, the patterns are presented more broadly as an abstract study of form. By including additional frequency components one might approach the level of irregularity seen in Figure 1. The use of natural coordinate systems for creating patterns in the plane is largely unexplored and has the potential to generate a seemingly unlimited variety of designs.

Acknowledgements

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