Circle patterns in Gothic architecture

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Abstract

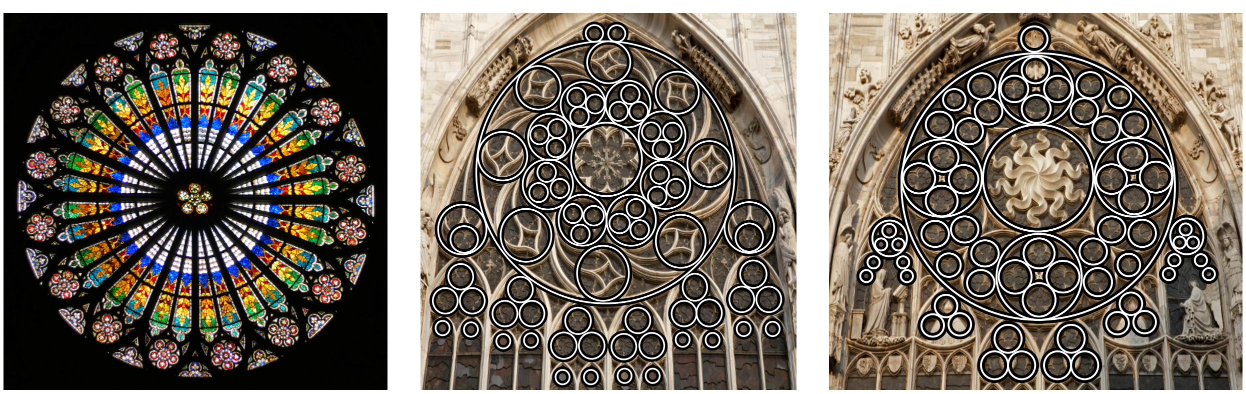
Inspired by Gothic-influenced architectural styles, we analyze some of the circle patterns found in rose windows and semi-circular arches. We introduce a recursive circular ring structure that can be represented using a set-like notation, and determine which structures satisfy a set of tangency requirements. To fill in the gaps between tangent circles, we add Appollonian circles to each triplet of pairwise tangent circles. These ring structures provide the underlying structure for many designs, including rose windows, Celtic knots and spirals, and Islamic star patterns.

1 Introduction

Gothic architecture, a style of architecture seen in many great cathedrals and castles, developed in France in the late medieval period [1, 3]. This majestic style is often applied to ecclesiastical buildings to emphasize their grandeur and solemnity. Two key features of Gothic architecture are height and light. Gothic buildings are usually taller than they are wide, and the verticality is further emphasized through towers, pointed arches, and columns. In cathedrals, the walls are often lined with large stained glass windows to introduce light and colour into the buildings.

In the mid-18th century, an architectural movement known as Gothic Revival began in England and quickly spread throughout Europe. The Neo-Manueline, or Portuguese Final Gothic, developed under the influence of traditional Gothic architecture and the Spanish Plateresque style [10]. The Palace Hotel of Bussaco, designed by Italian architect Luigi Manini and built between 1888 and 1907, is a well-known example of Neo-Manueline architecture (Figure 2).

In many of these Gothic-influenced architectural styles, circular arrangements were a prominent design motif. Rose windows are usually composed of multiple layers of rings of smaller circles inscribed within a larger circle. The circles may be tangent to or partially intersecting their neighbours. Some arrangements involve circles packed radially from the centre, forming spokes; these windows are also known as "wheel windows". Alternatively, the circular window may circumscribe some smaller circles (often arranged in a ring), each of which contains another set of inscribed circles, and so on. Figure 1 shows a wheel window from Strasbourg Cathedral and two rose windows from the Milan Cathedral, two famous examples of French Gothic architecture.



(a) (b) (c)

Figure 1: (a) An example of a wheel window from the Strasbourg Cathedral. (b, c) The Milan Cathedral has two different types of rose windows. The circular arragements found in them are traced in white. Photos from Shutterstock, reproduced with permission.

(a) (b)

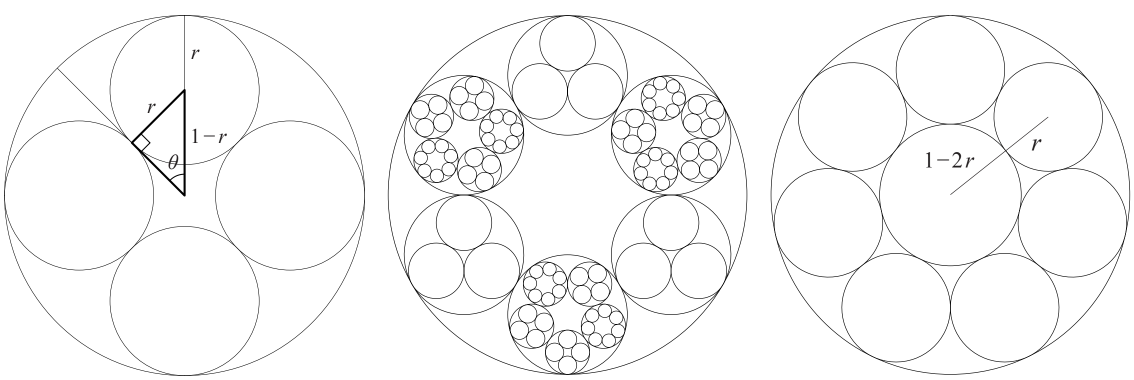
Figure 2: Circular arrangements found in (a) semi-circular arches, and (b) windows of the Palace Hotel of Bussaco are traced in white. Photos by Anita Chowdry, reproduced with permission.

Circular arrangements such as these are also found adorning semi-circular arches and windows in the Palace Hotel of Bussaco. In Figure 2, the circles are traced out, showing the rich variety of patterns used. Notice that sometimes (in the case of the arches), the larger circles can cut through the smaller circles rather than enclose them. Doing so makes the underlying circular arrangements less apparent and directs the viewer to other derivative shapes, such as the 4-pointed stars created by the four tangent circles. The pattern in Figure 2a also contains stray arcs and circles whose main purpose seems to be filling the gaps between other tangent circles. As these shapes have less obvious structures, we will not discuss them in this paper.

2 Construction of circular rings

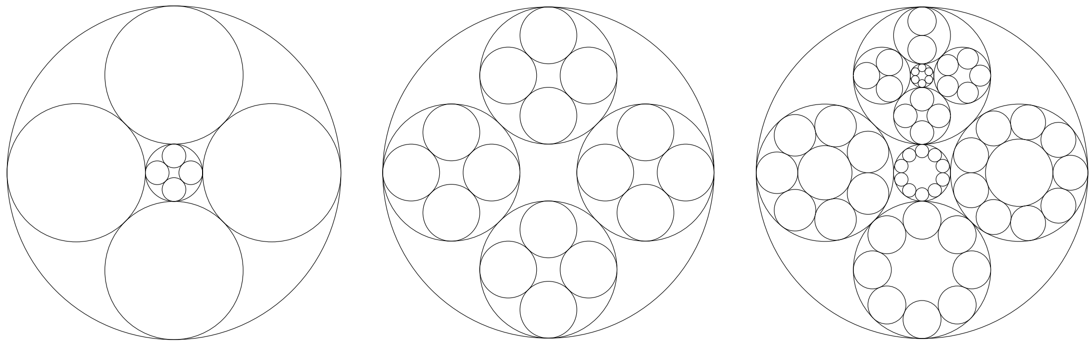
We are interested in generating circular arrangements similar to those found in Gothic-influenced architecture. The problem of finding planar circle packings given a set of circles has been studied extensively [8], but these deeply abstract mathematical concepts are most likely not used by architects, nor do they guarantee artistic results. Instead of exploring the space of all circle packings, we will focus on a very simple motif—the circular ring—found in many architectural examples discussed in Section 1.

Figure 3a shows a ring of size four inscribed in a larger circle. Let us briefly outline the method for constructing



(a) (b) (c)

Figure 3: (a) The diagram shows how to construct a ring of four circles (notation: $R(4)$). (b) We can recursively add rings to the smaller circles. (c) The extended notation adds a central circle inscribed by the ring (notation: $R(7;1)$).



(a) (b) (c)

Figure 4: The extended notation describes these circular rings as (a) $R(4;4)$, (b) $R(4^4)$, and (c) $R((2,3,4,5;6),(7;1),8,(9,1);10)$.

a ring of size n. Starting with the outer circle of radius 1 centred at the origin, we want to find r, the radius of the smaller n circles, such that they form a ring inside and tangent to the outer circle. We draw the triangle shown in Figure 3a whose vertices are the larger circle's centre, a smaller circle's centre, and the point at which that smaller circle is tangent to its neighbour. For a ring of size n, $\theta = \pi/n$. So we have $\sin\theta = r/(1-r)$, which simplifies to $r = 1/(1+\sin\theta)$. The ring can now be constructed by taking the circle with radius r centred at $(0, 1-r)$ and drawing copies of it rotated by $2i\pi/n$ for $i = 0, 1, \ldots, n-1$.

The ring construction can be extended by recursively adding rings to the smaller circles. We introduce a set-like notation to describe the more complex ring structures. Starting with the base cases, let $R(n)$ denote a circle inscribing a ring of size n (note that $R(1)$ refers to an empty circle). To increase the complexity, suppose for example that the inner circles of an $R(6)$ are replaced by alternating $R(3)$s and $R(5)$s; then the entire structure is denoted by $R(3,5,3,5,3,5)$, which we abbreviate to $R((3,5)^3)$ via exponentiation. If each $R(5)$ is then replaced by alternating $R(4)$s and $R(7)$s, then we write $R((3,(4,7,4,7,4))^3)$ (see Figure 3b). This notation can describe ring structures of arbitrary recursion depth. By convention, we start enumerating from the topmost circle and work our way around counter-clockwise.

As n increases, the circles in an $R(n)$ leave more empty space, which is undesirable because we want to decorate the whole circle. One way to resolve this issue is to use a concentric ring structure as shown in Figure 1b. In this case, however, the outer circles are not touching, so the spaces between them need to be filled with other decorative patterns. Another way to fill the empty spaces is to add smaller circles to the gaps, such as in Figure 1c, but this arrangement does not make use of the tangent circles in our ring structures.

To fill empty spaces using tangent circles, we propose an extension to our ring structure that includes a central circle tangent to the ring from the inside. The central circle can then be decorated recursively using the same ring structure. Our set-like notation extends naturally to incoporate a central circle. Let $R(n;1)$ refer to $R(n)$ with an inscribed central circle (see Figure 3c). In this notation, the part before the semicolon refers to the ring and the part after refers to the central circle. As an example, an $R(4)$ inscribed inside another $R(4)$ (see Figure 4a) would be written as $R(4;4)$. To leave out the central circle, simply omit the semicolon and what comes after it. For example, if we want only an outer ring containing $R(4)$s (see Figure 4b), we write $R(4,4,4,4) = R(4^4)$. Figure 4c shows a more complex example of the extended ring structure.

3 Calculating packing sequences

In Section 5, we will apply various rendering styles to the circular ring structures, but before doing so, we first need to determine which structures are suitable designs. Let us take another look at some of the architectural examples. In Figure 5a, each pair of tangent circles has a curve joining them through the tangent point. In Figure 5b, the most prominent shape in this semi-circular arch is the 8-pointed star, created by joining the arcs inside a ring structure formed by four $R(4)$'s. These examples demonstrate the importance of symmetry and the alignment of circles in the designs. We want to generate symmetric ring structures satisfying the tangency requirement that between any two

(a) (b) (c)

Figure 5: (a, b) In these architectural examples, tangent circles are connected through their interiors by curves. These curves are usually taken from the inscribed circular rings. (c) Our goal is to create ring structures satisfying the tangency requirement that between any pair of tangent circles, their inscribed rings are also tangent.

tangent circles in a ring, if they circumscribe any rings, then the inner rings are also tangent. We say two rings are tangent if the set of circles they contain touch at exactly one point, in this case, the tangent point of the two circles that enclose the rings (see Figure 5c).

To start, consider the structure given by $R(m^n)$. Given the value of n, we want to find the smallest integer m such that all the $R(m)$'s inscribed by tangent circles are mutually tangent. Figure 6a shows an example in which $n=7$ and $m=28$. Let the marked central angles in $R(n)$ and $R(m)$ be respectively θ_n and θ_m, which we know to be $2\pi/n$ and $2\pi/m$. Figure 6b shows a close-up of one of the $R(m)$'s. Dividing the circle up by the tangent points, the central angles created are $\pi-\theta_n$, $(\pi+\theta_n)/2$ and $(\pi+\theta_n)/2$. They can be rewritten respectively as $2(n-2)\phi$, $(n+2)\phi$ and $(n+2)\phi$, where $\phi=\pi/2n$. To satisfy the tangency requirement, each of these angles must be a multiple of θ_m, which means θ_m is their greatest common divisor (GCD). So we have[1]

$$\theta_m = \gcd(n+2,2(n-2))\phi = \gcd(n+2,-8)\phi = \begin{cases} \phi & \text{if } n\equiv 1,3,5,7 \bmod 8,\\ 2\phi & \text{if } n\equiv 0,4 \bmod 8,\\ 4\phi & \text{if } n\equiv 2 \bmod 8,\\ 8\phi & \text{if } n\equiv 6 \bmod 8,\end{cases} \tag{1}$$

which yields

$$m = \frac{2\pi}{\theta_m} = \begin{cases} 4n & \text{if } n\equiv 1,3,5,7 \bmod 8,\\ 2n & \text{if } n\equiv 0,4 \bmod 8,\\ n & \text{if } n\equiv 2 \bmod 8,\\ n/2 & \text{if } n\equiv 6 \bmod 8.\end{cases} \tag{2}$$

Next consider the same structure with a central circle given by $R(m^n;1)$. Figure 6c shows a close-up involving the measures of the angles created by the tangent points. A similar computation gives us the values or θ_m and m.

$$\theta_m = \gcd(n+2,n-2)\phi = \gcd(n+2,4)\phi = \begin{cases} \phi & \text{if } n\equiv 1,3 \bmod 4,\\ 2\phi & \text{if } n\equiv 0 \bmod 4,\\ 4\phi & \text{if } n\equiv 2 \bmod 4,\end{cases} \tag{3}$$

$$m = \frac{2\pi}{\theta_m} = \begin{cases} 4n & \text{if } n\equiv 1,3 \bmod 4,\\ 2n & \text{if } n\equiv 0 \bmod 4,\\ n & \text{if } n\equiv 2 \bmod 4.\end{cases} \tag{4}$$

As for the central circle, to satisfy the tangency requirement, the ring it contains must have a size divisible by n.

To distinguish between the cases with and without a central circle, let $m_R(n)$ and $m_C(n)$ respectively denote the minimum values such that $R(m_R(n)^n)$ and $R(m_C(n)^n;1)$ satisfy the tangency requirement. Note that the set of all values

[1]By Euclid's algorithm, $\gcd(a,b)=\gcd(a,ma+b)$ for any integer m. In this case, let $a=n+2$, $b=2(n-2)$ and $m=-2$.

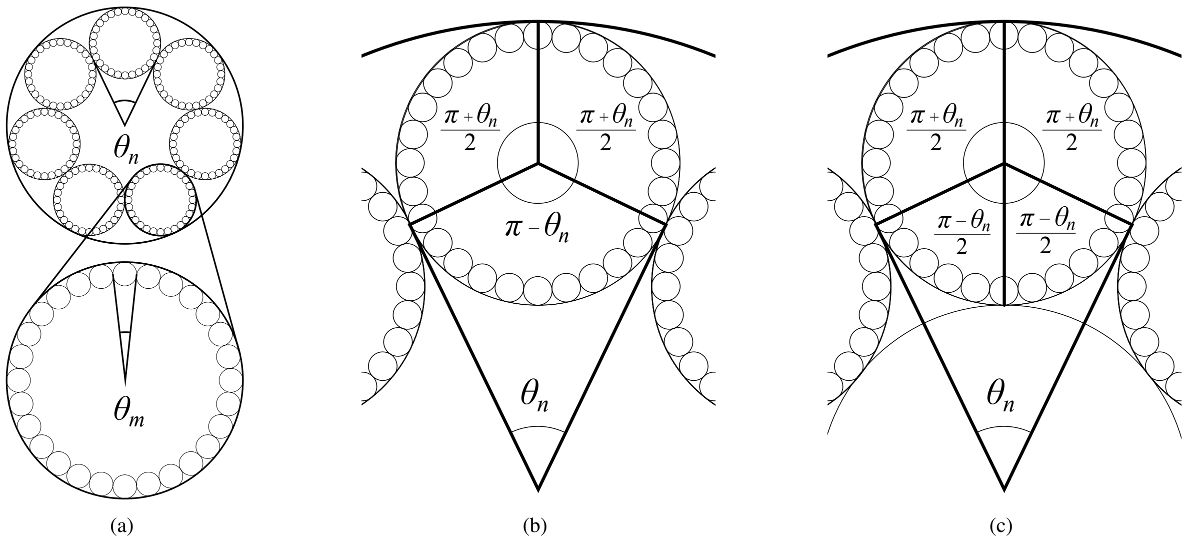


Figure 6: (a) In this example, the ring structure is given by $R(m^n)$ where $n = 7$ and $m = 28$. (b) To determine the minimum value of m that admits this configuration, consider the angles made by the tangent points. They must be integer multiples of θ_m. (c) The same is true for the case of $R(m^n;1)$, which involves a central circle.

of m such that $R(m^n)$ satisfies the tangency requirement is the set of all multiples of $m_R(n)$; the same is true for $R(m^n;1)$ with multiples of $m_C(n)$.

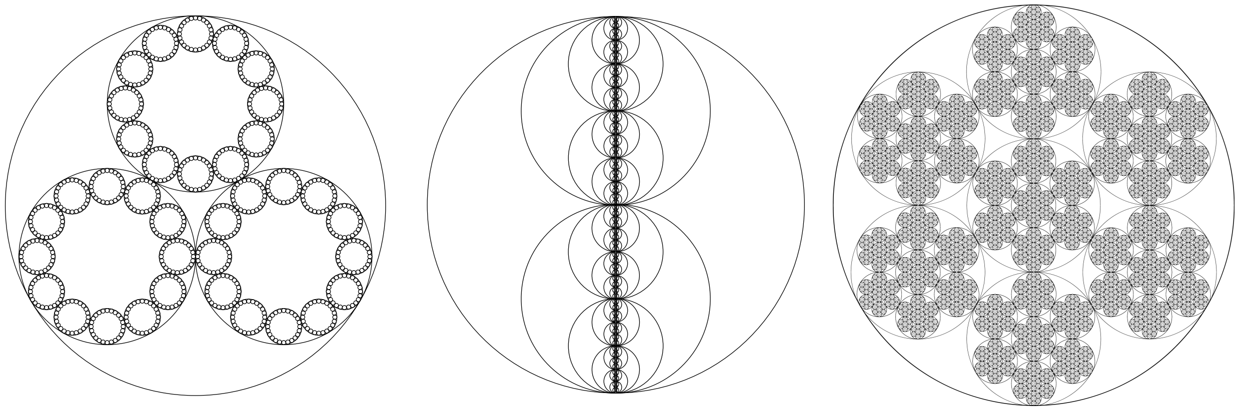
So far we have learned how to create two-level ring structures of the form $R(m_R(n)^n)$ and $R(m_R(n)^n;1)$ such that the smallest rings (i.e., the $R(n)$s) are arranged tangentially within the larger ring. For a ring structure with more than two levels, it satisfies the tangency requirement if between every two adjacent levels, the substructures satisfy the two-level tangency requirement. To illustrate this fact, consider the structure in Figure 7a given by $R((24^{12})^3)$. It satisfies the multi-level tangency requirement because the substructures $R(24^{12})$ and $R(12^3)$ both satisfy the two-level tangency requirement. In this case, we say that the sequence $\{3,12,24\}$ corresponds to a ring structure that obeys the tangency requirement.

Our goal is to find all sequences that obey the tangency requirement, and we can do so using the functions $m_R(n)$ and $m_C(n)$. Using the example from Figure 7a again, we know that this structure corresponding to the sequence $\{3,12,24\}$ obeys the tangency requirement because 12 is a multiple of $m_R(3)$ and 24 is a multiple of $m_R(12)$. In general, if x_{i+1} is a multiple of $m(x_i)$ ($m(\cdot)$ refers to either $m_R(\cdot)$ or $m_C(\cdot)$ here) for consecutive terms (x_i, x_{i+1}) in a sequence, then the tangency requirement is met. We want to use this property to find sequences that generate complex ring structures with nice aesthetic properties. Intuitively, we want long sequences containing relatively small numbers because they correspond to complex structures with circles that are not too small. Essentially we are looking for infinite sequences that satisfy the tangency requirement but do not diverge to infinity. These integer sequences can be either convergent or oscillating after a certain point.

Here is a formal statement of the problem: Let $\{x_i\}$ be a integer sequence that starts with $x_0 \geq 3$. Subsequent elements are defined as $x_{i+1} = c_i m(x_i)$ where c_i is some positive integer and $m(\cdot)$ refers to either $m_R(\cdot)$ or $m_C(\cdot)$ (we will discuss both cases). Enumerate all convergent and oscillating sequences $\{x_i\}$.

In the case where $m(\cdot) = m_C(\cdot)$, since $m_C(\cdot)$ is a non-decreasing function, there are no oscillating solutions, and the only convergent sequences are of the form $\{x_0, x_0, x_0, \ldots\}$, where $x_0 \equiv 2 \bmod 4$. In the case where $m(\cdot) = m_R(\cdot)$, the convergent sequences converge to some value congruent to 2 or 6 mod 8. As for the non-constant convergent sequences, let us assume for now that one exists. Consider the subsequence $\{x_i : x_i \equiv 6 \bmod 8\}$. It must be infinite because, according to the definition of m_R, $m_R(x_i) < x_i$ if and only if $x_i \equiv 6 \bmod 8$; if $\{x_i : x_i \equiv 6 \bmod 8\}$ were finite, then after some point, x_i would either converge or diverge to infinity, both of which contradict the assumption that $\{x_n\}$ is an oscillation.

Now it is possible to take two adjacent terms x_j and x_k (with $j < k$) from the subsequence such that $x_j = x_k$, otherwise $\{x_i\}$ diverges to infinity. It is also possible to make sure that $j+1 < k$, otherwise $\{x_i\}$ would be constant



(a) (b) (c)

Figure 7: (a) These ring structures correspond to the sequences $\{3, m_R(3), m_R(m_R(3))\} = \{3, 12, 24\}$, (b) $\{2, m_R(2), m_R(m_R(2)), \ldots\} = \{2, 2, 2, \ldots\}$, and (c) $\{6, m_C(6), m_C(m_C(6)), \ldots\} = \{6, 6, 6, \ldots\}$. In the rightmost structure, a central circle is included within each ring.

after some point. Our task then is to find the set of moves that takes x_j to x_k, or derive a contradiction in the process. Starting with $x_j \equiv 6 \bmod 8$, we have $m_R(x_j) \equiv 3, 7 \bmod 8$. $x_{j+1} = cm_R(x_j)$ for some positive integer c. If $c = 1$, then the next move necessarily takes us to $x_{j+2} \equiv 4 \bmod 8$ and then $x_{j+3} \equiv 0 \bmod 8$, which remains 0 mod 8 afterwards. If we choose $c = 2$, then $x_{j+1} \equiv 6 \bmod 8$ which means $k = j+1$; this contradicts with the assumption that $j+1 < k$. If we let $c \geq 3$, then $x_{j+1} > x_j$. In order to reduce the value back to x_j to get $x_k = x_j$, we need to visit another value congruent to 6 mod 8 before x_k, which contradicts the fact that x_j and x_k are adjacent terms in the subsequence $\{x_i : x_i \equiv 6 \bmod 8\}$. The contradictions found in all three cases suggest that oscillations do not occur.

In summary, for both $m(\cdot) = m_R(\cdot)$ and $m(\cdot) = m_C(\cdot)$, the only convergent sequences are constant sequences starting with $x_0 \equiv 2 \bmod 4$. Figure 7b shows the structure with no central circle corresponding a constant sequence of 2s. Figure 7c shows the structure corresponding to a constant sequence of 6s, in which the central circles are drawn.

Even though there are no oscillating infinite sequences, we can still find finite sequences with small numbers that produce interesting ring structures. The tangency requirement serves only as a guideline for choosing the sizes of the rings to work with. In Section 5, we will use ring structures that fully or partially satisfy the tangency constraints to test various rendering styles.

4 Filling the gaps with Appollonian circles

So far, we have taken the basic ring structure, added a central circle to fill the biggest gap, applied recursion to increase the complexity of the design, and determined which arrangements satisfy the tangency requirement. There are still some empty spaces left between the circles, and one way to fill these gaps is to add Appollonian circles [7] between triplets of pairwise tangent circles.

Descartes' theorem states that, for any triplet of pairwise tangent circles, there are two circles that are tangent to all three of them. These are known as the Appollonian circles. An Appollonian gasket (see Figure 8a) is constructed by taking such a triplet, drawing their Appollonian circles, and repeating this process for each new pairwise tangent triplet produced.

We can apply the idea of Appollonian circles to our ring structure. We start with $R(n;1)$ for some $n \geq 2$ which contains $2n$ tangent triples and draw all the Appollonian circles. With this approach, some circles intersect, which produce intricate patterns (see Figure 8b) but are not the type of circle packing we are looking for. To remove the unwanted intersections, we pose the additional constraint that for each pairwise tangent triplet, an Appollonian circle is only drawn if it does not circumscribe the original three circles. The revised drawing method produces the type of result shown in Figure 8c.

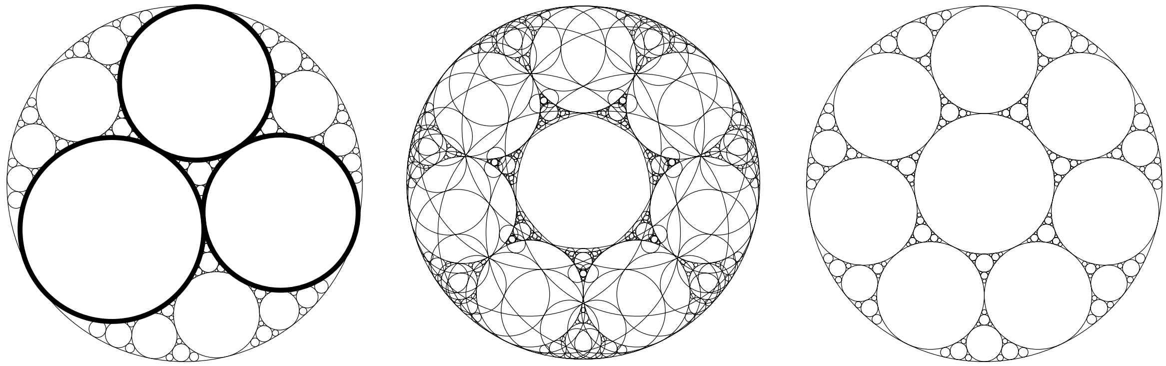


Figure 8: (a) A regular Appollonian gasket is generated from three pairwise tangent circles. Applying this concept to our ring structures, instead of (b) drawing all the Appollonian circles and produce unwanted intersections, (c) we draw only an Appollonian circle if it does not circumscribe the pairwise tangent triplet from which it is derived.

5 Experimenting with different rendering styles

The combination of circular ring structures and the Appollonian circles allows us to create a wide range of designs. Figure 9a shows the underlying circle pattern used for the rose window design in Figure 10a. As seen in the Neo-Manueline arches, the border cuts through the ring structure along circle centres and tangent points.

Even though the underlying structure consists of circles, we explore other decorative possibilities by replacing each circle with a circular design. The propeller pattern in Figure 9b is used in a 2-level hexagonal ring structure to create the tattoo-like design in Figure 10b.

Gothic architecture draws elements from the elaborate ornamental designs of Islamic art [5]. We looked into traditional Islamic art and found that many designs contain circular structures [6]. To create our artwork, we generated a ring structure composed of $R(6)$s and $R(12)$s, replaced them with 6- and 12-pointed stars respectively (see Figure 9c), added 5-stars in the gaps, and combined everything to form the star pattern shown in Figure 10c.

Celtic art also uses circular elements in many designs, such as knots, spirals and key patterns [2]. To construct the spiral design shown in Figure 10d, the circles in a ring structure are replaced by the simple spiral shown in Figure 9d. As for creating a Celtic knot from a ring structure, we first took a regular circle knot (see Figure 9e, top) and cut some strands so that it can be joined up with another identical knot. Then six such knots are joined in a circle to produce the knot shown in Figure 10e. Other methods for constructing circular knotworks have been developed by Åström and Åström [9]. Leonardo Da Vinci also designed some intricate knotworks involving concentric circle patterns [4].

In some of the architectural examples, circle patterns are interlaced to create more intricate designs. In Figure 10f, four ring structures of different sizes are superimposed to create a complex pattern.

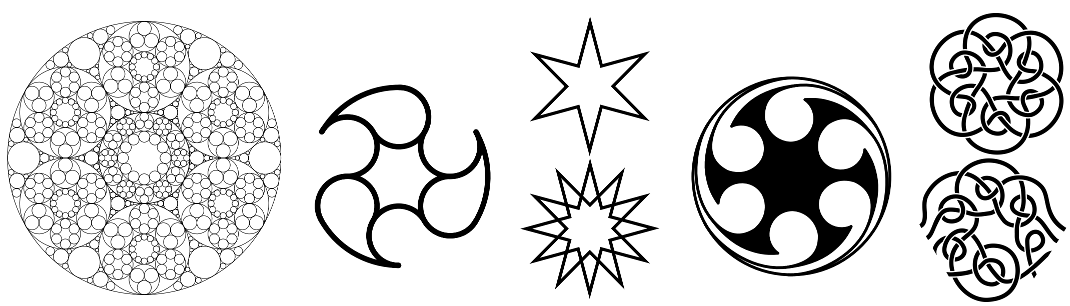


Figure 9: Design elements used to create the artworks in Figure 10

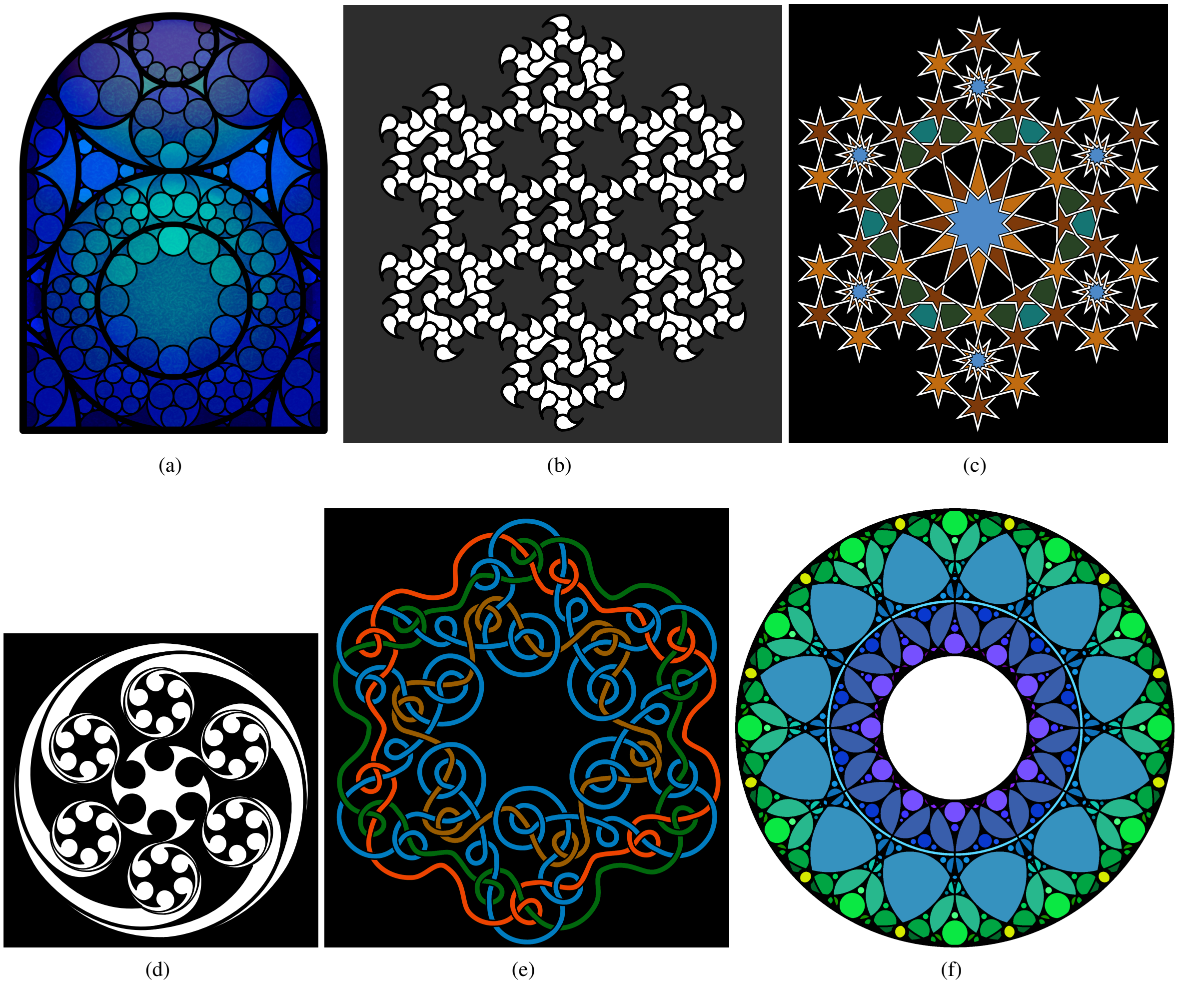


Figure 10: These designs are created from the design elements in Figure 9 with underlying ring structures.

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