Folding Your Way to Understanding*

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Abstract

The goal of this paper is to describe how modular origami can be utilized in a geometry classroom not only to enhance student understanding of two-dimensional concepts, but also as a tool for three-dimensional problem solving. All participants in this workshop will create a Sonobe unit and explore the relationship between the original paper and the final folded unit. Some workshop participants will create more Sonobe units and put them together to form polyhedra. The other participants will be charged with finding the volume of the polyhedra that have been created.

1. Introduction

Origami is an activity that is deeply based in geometry, yet it is rarely used in geometry classrooms. Students can be instructed on how to fold a piece of paper using geometric terminology—a great way to assess whether students can understand and apply these terms. After folding thirty pieces of paper into what is known as a Sonobe unit, a modified version of a Platonic solid (an augmented icosahedron) can be assembled. By presenting students with a completed model of this solid and providing them with the materials to make it, students will use problem-solving skills to recreate the three-dimensional figure. Once constructed, students can then be presented with the challenge of finding the volume of the solid. While this may seem like a daunting task, the augmented icosahedron can actually be decomposed into two sets of twenty congruent pyramids. The dimensions of these pyramids can then be found by utilizing ideas spanning from simple geometric relationships to the golden ratio, making this activity a hands-on, problem-solving experience applicable to a wide range of ages and abilities.

2. Building the Augmented Icosahedron

2.1 Creating a Sonobe Unit Some origami involves one piece of paper. Modular origami involves many pieces. In order to create an augmented form of a twenty-sided, regular, three-dimensional figure (a regular icosahedron), thirty identically folded pieces of paper are needed. The resulting folded design is called a Sonobe unit. During the workshop, each participant will make at least one Sonobe unit. To make a Sonobe unit, begin with a square piece of paper. Fold it in half to form two congruent rectangles. Open the square and fold the sides parallel to the first fold to the center to create four congruent rectangles. Open the square so the rectangles are side-by-side, not on top of each other. Fold the top left corner to the closest of the parallel folds to form a 45-45-90 right triangle [See Figure 1]. Then, fold the hypotenuse of this triangle to the same parallel fold [See Figure 2]. Rotate the paper 180°, and repeat this process to what is now the top left corner. Refold the outer rectangles to the center and hold the paper so it is longer

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horizontally than it is vertically. The folds should make a "z" shape [See Figure 3]. Fold the lower right corner to the top of the rectangle to form a 45-45-90 right triangle and tuck the right angle underneath the fold. Rotate the paper 180°, and repeat this process on what is now the bottom right corner. Now it is a parallelogram [See Figure 4]. Turn to the side where none of the folds are visible and fold an acute angle to the obtuse angle that is farthest from it [See Figure 5]. Repeat this for the other acute angle. With the resulting square, fold backwards along the visible diagonal, then open the last three folds [See Figure 6]. This is one piece of the icosahedron—one Sonobe unit. When folding all thirty of these figures, it is crucial that they are all folded the exact same way. To check this, each folded piece should stack perfectly on top of the others. If this is not the case, the icosahedron cannot be built.

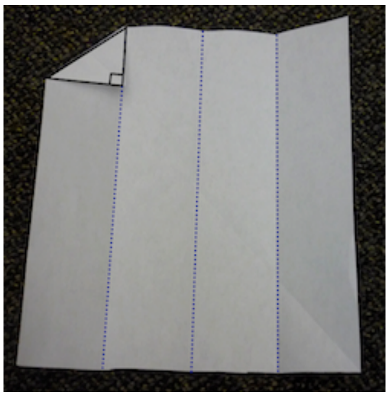


Figure 1: *Folding the top corner into a 45-45-90 triangle*

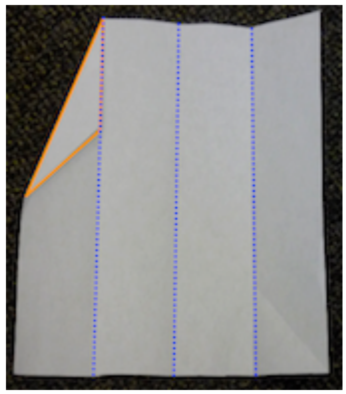


Figure 2: *Folding the hypotenuse to the parallel fold*

Figure 5: *Folding the acute angle to the obtuse angle along the same edge*

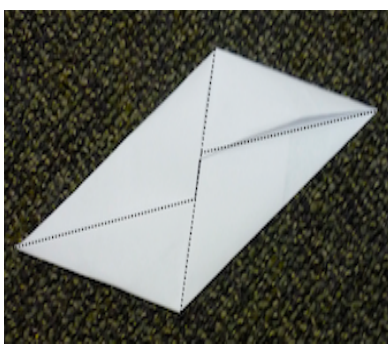


Figure 4: *The parallelogram*

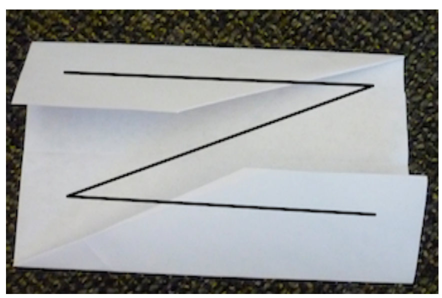


Figure 3: *The "z-shape" fold*

2.2 Piecing the Icosahedron Together At this point in the workshop some participants will continue to this step and others will move to section 3. To build the augmented icosahedron, a few terms must first be defined. Each Sonobe unit has four distinct triangular sections, each congruent in size. On either end are "tabs;" in between the tabs are two "pockets" [See Figure 6]. Notice that the tab from one piece fits perfectly into the pocket of another. During construction, each piece should always be kept with the pockets facing upward (or on the outside of the soon-to-be icosahedron).

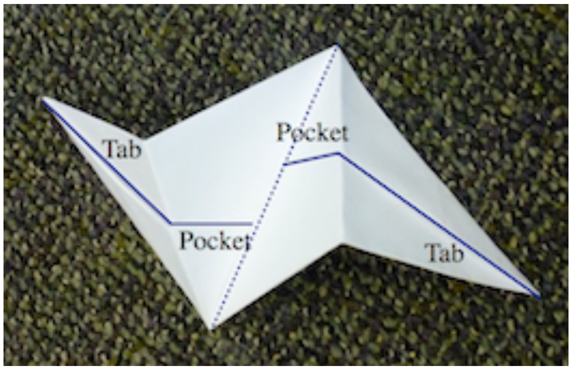


Figure 6: *The completed Snoobe unit, ready to be used to make an augmented icosahedron*

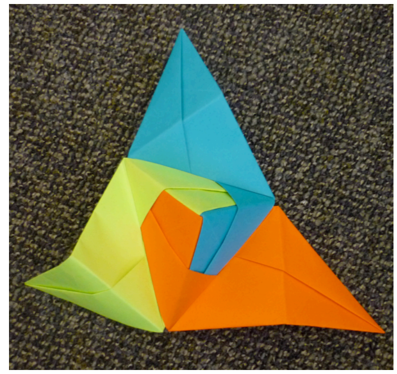


Figure 7: *One completed stellation*

To begin, it is fairly simple. Take a tab of one Sonobe unit and insert it snugly into a pocket of a second unit. Now, take a third Sonobe unit and insert its tab fully into to a pocket of the second unit that is adjacent to the tab inserted into the first unit's pocket. Next, take the first unit's tab that is adjacent to its filled pocket and insert the tab into the pocket on top of the third unit that is adjacent to the tab that was inserted in the previous step. Congratulations, one of the twenty pyramidal augmentations required for this construction has been completed [See Figure 7]. Notice that this pyramid has an equilateral triangle as its base, which is one of the 20 faces of the icosahedron. Now, make another pyramid beginning with an empty pocket of one of the three Sonobe units you have already used and two unused Sonobe units. Then, make a third pyramid just as the second was formed.

Observe that all three pyramids made thus far each have one slant edge that meet at the same point. This point is one of the twelve vertices of the icosahedron. The key to building the augmented icosahedron is to keep in mind that at every vertex on an icosahedron, five triangular faces meet. Because these faces are the bases of the pyramids, this means that five pyramids must meet at each vertex. To do this, construct a fourth pyramid just as the second and third pyramids were made, but be sure to form this pyramid so that one of its slanted edges meets at the same vertex as the other three. If the bases of the four pyramids are laid flat on a table, it can be observed that the unused tab from one of the pyramids is beginning to overlap an empty pocket of another piece [See Figure 8]. To form the fifth pyramid for this vertex, insert the overlapping tab into this empty pocket, and then use another Sonobe unit to complete this pyramid.

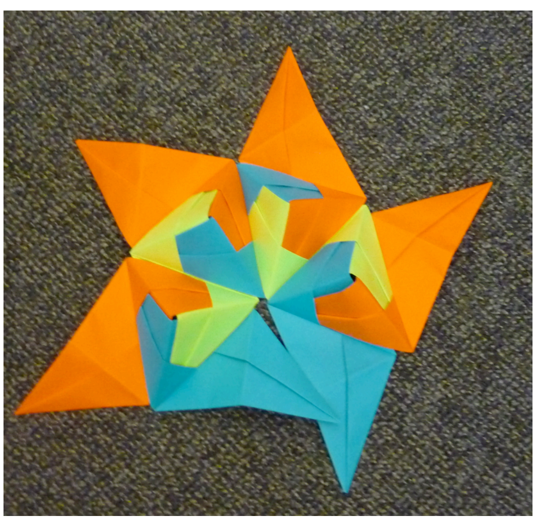


Figure 8: *Four pyramids with an overlapping tab and pocket; this shows that the fifth pyramid must be made*

Now, flip the five connected pyramids over so that their tips are resting on the table. Notice that a pentagon, whose edges are formed from a side of each base of the five pyramids, can be seen [See Figure 9]. This pentagon will be important in Sections 4 and 5 of this paper. To complete the rest of the origami figure, notice that at each vertex of the pentagon, two pyramids meet. The vertices of the pentagon are also vertices of the icosahedron, and three more pyramids must meet at each vertex in order to complete the icosahedron [See Figure 10].

In the end, all thirty Sonobe units should have been utilized, and there should be neither any empty pockets nor any unused tabs of any of the Sonobe units. It is easy for a tab to slip behind a pocket and end up on the inside of the icosahedron. Once the augmented icosahedron is complete, it is time to decompose the figure in order to find its volume.

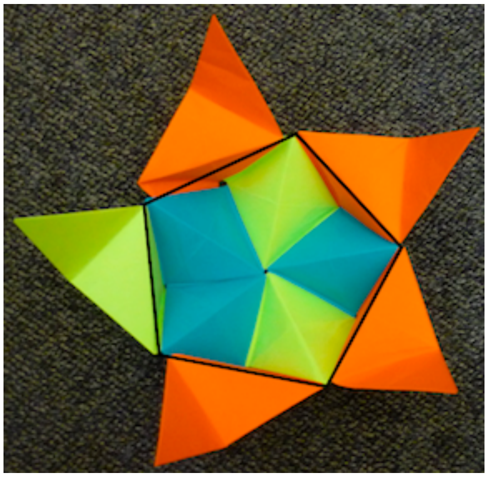


Figure 9: *The interior pentagon made by the pyramids*

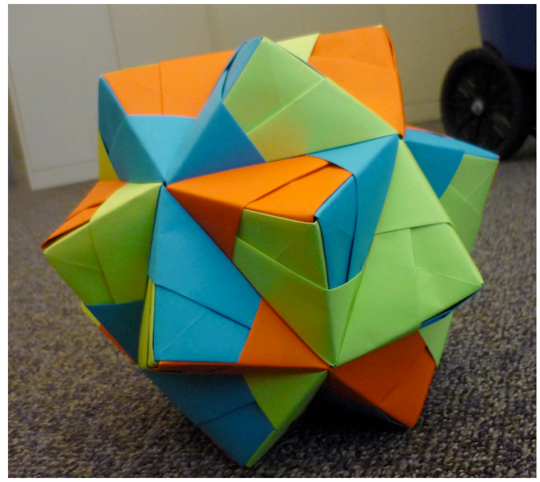


Figure 10: *The completed augmented icosahedron*

2.3 Other Modular Origami Constructions Using Sonobe Units While our workshop focuses on icosahedra, there will be a discussion of other forms that can be created using Sonobe units. To construct the augmented icosahedron, thirty Sonobe units were used to create groups of five pyramids around each vertex. What is the result if four pyramids are placed around each vertex? The participants will find that because each pyramid has an equilateral triangle as a base, if four pyramids are placed around a vertex, an augmented octahedron is formed. This construction will require twelve Sonobe units.

A mathematical way to see how many Sonobe units will be required to construct one of these figures is to consider how many faces the original solid has. For example, an octahedron has 8 faces and an icosahedron has 20 faces. Each triangular pyramid creates three faces for every original face of the icosahedron, and each Sonobe unit covers two of these new faces. Therefore, the number of Sonobe units required to construct one of these solids is $\frac{3}{2}$ times the number of original faces of the polyhedron. This results in 30 required Sonobe units for the augmented icosahedron and 12 Sonobe units to build the augmented octahedron.

What will happen when six pyramids are formed around each vertex? Again, the participants will see that this result depends upon the fact that the base of each pyramid is an equilateral triangle. When six equilateral triangles converge at one point, a total of 360° are placed around the point. Therefore, this arrangement results in an augmented tessellation. It should be noted that because this tessellation does not form a distinct solid (because it can continue infinitely), the aforementioned formula to find the required number of Sonobe units to construct the tessellation cannot be applied.

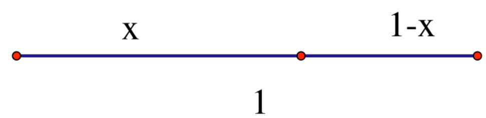
3. Decomposition of the Augmented Icosahedron in Order to Find its Volume

3.1 Finding Volume The augmented icosahedron can be created to help students understand more about polyhedra. Students can be presented with the task of finding the volume of an augmented icosahedron. The first step to finding the volume of this figure is to decompose it into simpler shapes. During the workshop, some participants will work together to find the volume of the icosahedron. After attempting to find the volume on their own, the workshop participants will be presented with the information below.

3.2 Augmentations Compare the model of the augmented icosahedron to a picture (or a model) of a regular icosahedron. It can be seen that the only thing that makes the augmented icosahedron distinct from the regular icosahedron is, obviously, the pyramidal augmentations. Therefore, if each of these augmentations were cut off the figure, we would have twenty congruent pyramids and a regular icosahedron.

3.3 Icosahedron Now that the obvious decomposition has been made, it is time to decompose the icosahedron itself. To do this, begin by finding the center of each face. On the augmented icosahedron, finding the center is easy—it is directly underneath the apex of each pyramid. Then, by constructing a line through the center that is perpendicular to each face, it can be seen that all of these lines meet at the center of the icosahedron [2]. To show students where the center is on the origami augmented icosahedron, something rigid, such as a piece of uncooked spaghetti or a narrow dowel rod, can be placed through the apex of each pyramid so that it comes out of the apex of the pyramid on the opposite side of the figure. If this is done with two or three sets of opposite faces simultaneously, it can be seen that all of these lines meet in the same place—the center of the icosahedron. We will call this point O.

4. Pentagons and the Golden Ratio



$$\frac{1}{x}=\frac{x}{1-x}=\phi$$

Figure 11: *A line segment divided into the golden proportion*

Often in mathematics we find unexpected relationships between numbers. One of these relationships is the golden ratio. Known as phi, ϕ, the golden ratio is the relation of a part to the whole such that it will equal the ratio of the part to the remainder of the whole. In Figure 11, the line segment of length 1 is divided into two segments—one of length x, and the other of length 1-x. The value of the golden ratio can then be found by cross-multiplying the equivalent proportions $\frac{1}{x}$ and $\frac{x}{1-x}$, which yields the quadratic equation $f(x)=x^2-x-1$. This equation has roots at $\frac{1\pm\sqrt{5}}{2}$, which means $\phi=\frac{1+\sqrt{5}}{2}\approx 1.618$. The golden ratio is found around the world and in many different subjects, but is also a key part of the regular icosahedron.

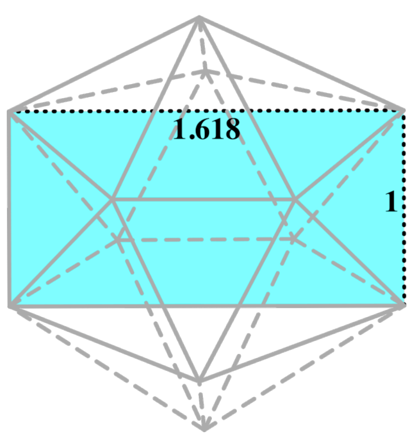


Figure 12: *A golden rectangle inside the icosahedron*

In two dimensions, the golden ratio is closely related to pentagons. When the non-adjacent angles of a pentagon are connected, the line segments form a pentagram, and each line segment is split into the golden proportion by each intersecting line segment. How does this connect to icosahedra? The answer lies in the layout of the faces.

As previously stated, each vertex of the regular icosahedron is the meeting point of five equilateral triangles. When connected, the sides not touching the other triangles join to form a regular pentagon. As this relationship occurs at every corner, the edges of the icosahedron form pentagons [See Figure 9] [2].

Because of this relationship, we can form golden rectangles within the icosahedron. A golden rectangle is a rectangle in which the ratio of the short side to the long side is the golden ratio. If we take each edge of the icosahedron to be the short side of a rectangle and extend the rectangle through point O to meet an edge on the other side of the icosahedron, a golden rectangle is formed [See Figure 12] [1]. This fact is necessary for finding the exact volume of an icosahedron, but workshop participants are not expected to realize this on their own.

5. Finding the Volume of the Augmented Icosahedron

5.1 Finding the Volume of an Augmentation It has been mentioned previously that the augmentations are simply twenty congruent pyramids with triangular bases. Therefore, the volume of only one of these pyramids must be found, and can then be multiplied by twenty. The volume of the pyramid can be found by the formula $V = \frac{1}{3}Ah$ where A represents the area of the base and h represents the height of the pyramid. Since the base is an equilateral triangle with side length s, its area is $\frac{\sqrt{3}}{4}s^2$ [2].

Finding the height of the pyramid is slightly more complex. First, by carefully observing (or marking) and then unfolding one of the pieces used to construct the augmented icosahedron, it can be seen that s is equal to half the length of the side of the paper used to make the piece. Furthermore, the edge length of the pyramid, e (the length from the base vertex to the apex), is represented by the line segment $\overline{FJ}$, shown in Figure 13. By the properties of the folds used to make the piece, it can be seen that EFHG is a square whose side length is half the length of the side of the paper—making its side length s. So line segment $\overline{FG}$, the diagonal of EFHG, has a length of $s\sqrt{2}$. Because a square's diagonals bisect each other, $\overline{FJ}$ then has a length of $\frac{s\sqrt{2}}{2} = e$. By applying the formula $e = \sqrt{h^2 + \frac{1}{3}s^2}$ and substituting for e, it is found that $h = \frac{1}{\sqrt{6}}s$ [3]. Thus, the volume of one of the pyramids can be found using the equation $V = \frac{1}{3} \cdot \frac{\sqrt{3}}{4}s^2 \cdot \frac{1}{\sqrt{6}}s$, which reduces to $V = \frac{1}{12\sqrt{2}}s^3$. Because there are twenty of these pyramids, the total volume for all the augmentations is found by the expression $V_{\text{Augmentations}} = \frac{5}{3\sqrt{2}}s^3$.

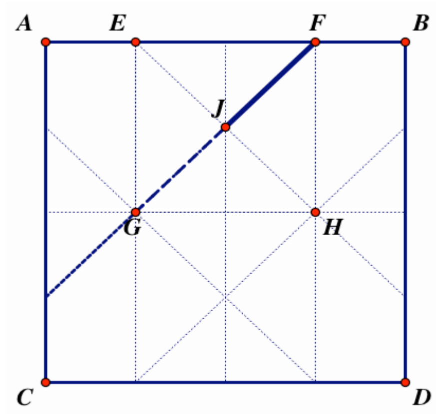


Figure 13: *A scaled representation of key folds made to each piece of the augmented icosahedron.*

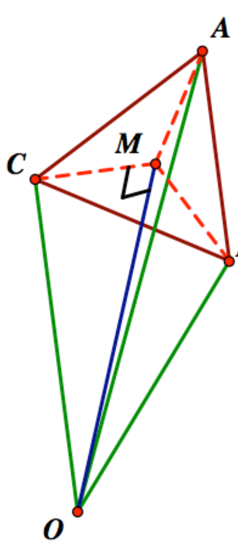


Figure 14: *Representation of pyramid within icosahedron*

5.2 Finding the Volume of the Icosahedron An icosahedron is not a solid whose volume is easily found algebraically. In order to find the volume, it is crucial to see that each face of the icosahedron forms the base of a pyramid with vertex at the center (point O) of the icosahedron. Because there are twenty congruent faces, there are twenty congruent triangular pyramids. This conceptualization can be modeled with zometools, and will be shown to participants.

As was discussed earlier, the formula for the volume of a triangular pyramid is $V = \frac{1}{3}Ah$ where A is the area of the base and h is the height of the pyramid. It can be seen in Figure 14 that $\overline{OM}$ is the height of the pyramid [1]. The base is one of the equilateral triangle faces of the icosahedron, ΔABC. It is known that the area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$ [1]. When the icosahedron is pictured inscribed in a sphere, each vertex of the icosahedron will be on the surface of the sphere and O would be the center of the sphere. Because points A, B, and C are corners of the icosahedron, the segments $\overline{AO}$, $\overline{BO}$, and $\overline{CO}$ are all radii of the sphere. Furthermore, because $\overline{OM}$ was constructed to be perpendicular to the base and through the center of the triangle, ΔAMO, ΔBMO, ΔCMO are congruent right triangles [See Figure 14] [1]. This can be used to find the height of the pyramid, which is the length of $\overline{OM}$ [See Figure 14]. Focusing on ΔCMO, it is known that $CM = \frac{s}{\sqrt{3}}$. Now it is necessary to find the radius, *CO,* in terms of *s*. Doing this requires using the aforementioned golden ratio and pentagons.

Any line through a vertex of the icosahedron and the center, O, will pass through another vertex. Because all of the vertices lay on the sphere, each line segment from vertex to vertex through O is a diameter of the sphere. As was previously discussed, each edge of the icosahedron forms a golden rectangle within the icosahedron [See Figure 15] [1]. These diameters are also the diagonal of the golden rectangle [1]. Each edge of the icosahedron is the side of a pentagon. In Figure 15, $\overline{CB}$ and $\overline{DE}$ are edges of the icosahedron and sides of a pentagon. This makes $\overline{CE}$ the diagonal of the pentagon. The length of the diagonal of a pentagon is the length of the side of the pentagon multiplied by ϕ [2]. By substitution, the diagonal of a pentagon is $\frac{\sqrt{\phi^2+1}}{2}s$. Using the Pythagorean theorem, it is found that $CD^2 + DE^2 = CE^2$. This becomes $(\phi s)^2 + s^2 = CE^2$ which leads to $CE = s\sqrt{\phi^2+1}$. Now that the length of the diameter has been found, it follows that the length of the radius is $CO = \frac{\sqrt{\phi^2+1}}{2}s$ [2].

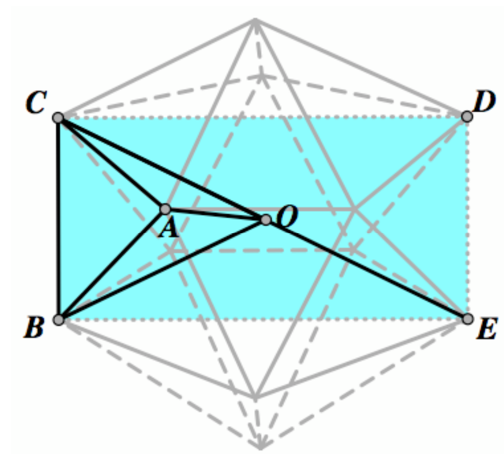


Figure 15: *The golden rectangle located within the icosahedron*

Now the height of the pyramid within the icosahedron can be found. Since ΔCMO is a right triangle, the Pythagorean theorem yields, $CO^2 - CM^2 = OM^2$ which becomes $OM^2 = \left(\frac{\sqrt{\phi^2+1}}{2}s\right)^2 - \left(\frac{1}{\sqrt{3}}s\right)^2$. Ultimately, $OM = \frac{\phi^2}{2\sqrt{3}}s$. Now that the height of each internal pyramid has been found, the participants in the workshop can find the volume of the icosahedron!

The formula for the volume of a pyramid is $V = \frac{1}{3}Ah$, where *A*, the area of the equilateral triangular base, is equal to $\frac{\sqrt{3}}{4}s^2$ and *h* is $\frac{\phi^2}{2\sqrt{3}}s$. Thus, the formula for the volume of one internal pyramid within an icosahedron with edge length *s* is $\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right)\cdot\left(\frac{\phi^2}{2\sqrt{3}}s\right)$ which simplifies to $\frac{\phi^2}{24}s^3$. There are twenty faces of an icosahedron, each forming the base of one internal pyramid. Therefore, in order to find the volume of the entire icosahedron, this value must be multiplied by twenty, yielding $V_{\text{Icosahedron}} = \frac{5\phi^2}{6}s^3$.

5.3 Finding the Volume of the Augmented Icosahedron The volume of all twenty pyramids and the icosahedron itself have now been found. The final step is to combine these volumes to find the volume of the entire shape, the augmented icosahedron. To do this, we simply add the volumes we have $V_{\text{Augmented Icosahedron}} = V_{\text{Augmentations}} + V_{\text{Icosahedron}} = \left(\frac{5}{3\sqrt{2}}s^3\right) + \left(\frac{5\phi^2}{6}s^3\right)$. Recall that s is equal to one half of the side length of the paper used to create a Sonobe unit. This means that the volume of the augmented icosahedron can be found before folding a single piece of paper.

5.4 A Few Words About Precision While the calculations above can be used to find the precise volume of the augmented icosahedron, to have participants of this activity (whether at a workshop or in a classroom) derive these formulas in a short amount of time after folding a few pieces of paper is unreasonable. Participants in this activity can find the area of each face of the icosahedron and the heights of each pyramid relatively simply with only a ruler and a piece of string. Once participants have approximated the volume of this solid in this way, they could then be given the formula to calculate the precise volume of the solid and be asked to find their percent error of the volume, which is found by the formula $\frac{\text{(Approximated volume)} - \text{(Precise volume)}}{\text{Precise volume}}$. The formula for the precise volume of the figure can either be given to students directly, or more advanced students could be asked to derive or simplify parts of it.

6. Conclusion

At the end of the workshop both groups will be brought together to display their findings. The folding group will share the different polyhedra they were able to create and the volume group will explain the process of calculating the volume of an augmented icosahedron. Beginning with a few simple geometric folds, students can create a complex formation. Exploring properties of this formation, including its volume, will lead students to improve their problem-solving skills. This origami model can help students understand the workings and visualizations of regular polyhedra. While some of the formulas become complicated, most were derived using knowledge of two-dimensional figures and the Pythagorean theorem. Origami is rarely used in classrooms but this problem- solving activity demonstrates its ability to help students visualize and understand the geometry they have learned.

References

[1] Knott, R. (2009). *The golden geometry of solids or Phi in 3 dimensions*. Retrieved from http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi3DGeom.html

[2] MacLean, K. *The icosahedron*. Retrieved from http://www.kjmaclean.com/Geometry/Icosahedron.html

[3] Weisstein, E. (2010). *Triangular pyramid*. Retrieved from http://mathworld.wolfram.com/TriangularPyramid.html