A Workshop on Stellation Inspired Sculpture

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Abstract

In this workshop we will study the structure of some of the complex and beautiful sculptures by George Hart. Knowledge of the geometric concept of stellation is fundamental to understanding how these sculptures are designed and constructed. We will learn how stellation is used to produce complex polyhedra. The author has designed a simple puzzle sculpture kit based on this concept. Each participant will assemble one of these kits. Constructing and studying this small sculpture will enhance the understanding and appreciation of George Hart's creations.

Figure 1: *The Small Ball of Fire sculpture*

1. Introduction

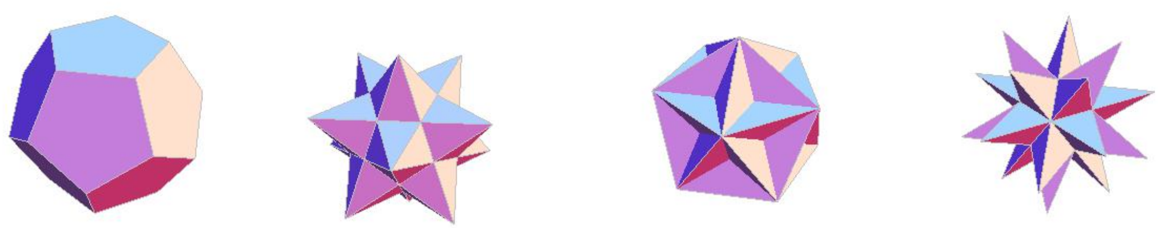
George Hart is well known for his creative geometric sculptures in metal, wood, plastic, and paper. Many of his works are based on stellations of uniform polyhedra. Dr. Hart generously shares his design concepts on his website [1] and in his publications [2][3][4]. In [3] he discusses several designs for laser cut acrylic, wood, and metal puzzles that are small versions of his sculptures. He also mentions using selective laser sintering of nylon to make more flexible puzzle pieces.

The puzzles discussed in [3] are not for beginners and most are extremely difficult. In [4] Dr. Hart notes that "considerable skill in assembly is needed—they are much harder to build than they look. In fact, I wonder if anyone but me could assemble all of these designs." He later wagers a beer if anyone can assemble his *Eights* puzzle design [4]. After experimenting with Hart's designs using paper I wanted to create a simplified version of this type of puzzle that would make the underlying concepts comprehensible to a wide audience. I settled on craft foam as a material since it is cheap, available, and easy to cut. It is more permanent than paper and holds up well to bending and repeated assembly. I also chose to make the puzzle multicolored to make the structure easy to see. While Dr. Hart's goal is to make puzzles that are challenging my goal is to share a basic understanding of this type of structure and to increase appreciation for the complexity of Hart's sculptures.

2. Geometric Concepts

Given a polyhedron another polyhedron can be formed by extending the faces until the extensions intersect [5]. The resulting polyhedron is called a stellation. If the original polyhedron is uniform and all its faces are identical the stellation can be thought of being constructed of identical polygons (with many intersections) glued to the faces of the original polyhedron. The simplest example of a stellation is the stella octangula, which is formed by extending the faces of the octahedron. The stella octangula is also the compound of two tetrahedra. The triangular faces of these two tetrahedra can be perceived as intersecting triangular faces glued to the smaller triangular faces of the octahedron. Extending the faces of the octahedron farther does not lead to any other intersections so the stella octangula is the only stellation of the octahedron.

The tetrahedron and the cube have no stellations since the extensions of their faces never intersect. The dodecahedron has 3 stellations [5]. The first is called the small stellated dodecahedron and is made of twelve pentagrams (five pointed stars) glued to the faces of the dodecahedron. The extensions of the faces of the dodecahedron intersect two more times to form the great dodecahedron and the great stellated dodecahedron (see figure 2). The extensions of the faces of the icosahedron can be configured into 59 different uniform polyhedra [6]. The rhombic triacontahedron has 227 stellations [7].



(a) (b) (c) (d)

Figure 2: *The (a) dodecahedron, (b) small stellated dodecahedron, (c) great dodecahedron, (d) great stellated dodecahedron. Faces that are the same color are extensions of the same face of the dodecahedron.*

When designing a sculpture based on a polyhedron stellation, we must carefully consider all the intersections of the face plane extensions. The sculpture pieces can intersect at these lines or be designed to share the extended face without intersecting. Avoiding intersections creates an open woven structure. The more face plane intersections there are, the more complex the design must be to avoid intersections. For example, George Hart's sculpture *Compass Points* [1] is based on the 15^{th} stellation of the rhombic triacontrahedron and is thus very complex.

The sculpture we will construct is called *The Small Ball of Fire* since it is based on the small stellated dodecahedron and the colors and shape are reminiscent of fire. The face planes of the small stellated dodecahedron intersect only twice: at the edges of the dodecahedron and again at the edges of the stellation. I designed the pieces to curve past each other at the edges of the dodecahedron and then to slide into intersecting notches at the second intersection. The slots for attaching the pieces lie on the edges of a pentagram since the small stellated dodecahedron is constructed from 12 pentagrams.

3. The Sculpture Kit

The sculpture kit consists of 12 identically shaped pieces die cut from 2mm craft foam. There are 3 pieces in each of 4 colors: red, green, yellow, and orange. A template of a piece is shown in figure 3 and pieces can be hand cut from 2mm craft foam using this template. The template should be enlarged so that the slots are 2mm wide and the edges of the pieces fall on a circle of diameter of 7.5 inches. Precut kits with color instructions can be ordered from Tesselations, Inc. at www.mathartfun.com.

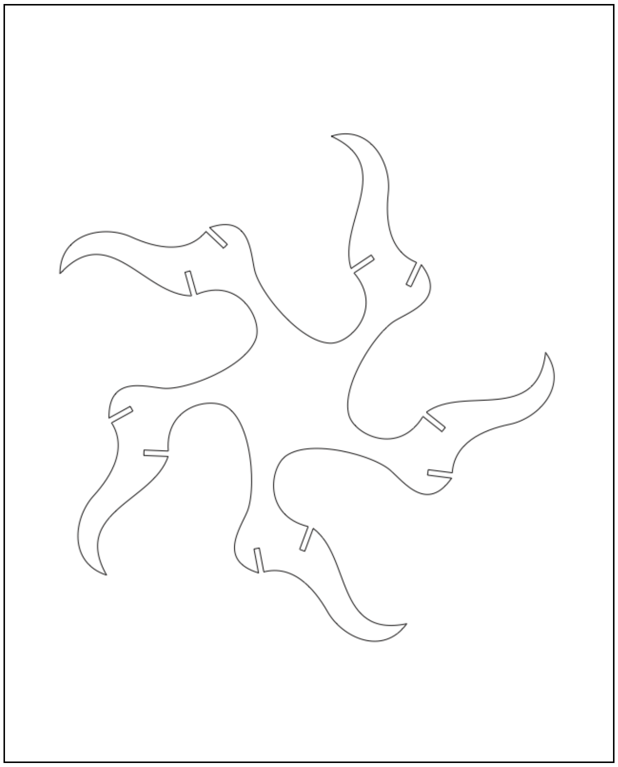


Figure 3: *Sculpture piece template*

4. Assembly Instructions

Each piece of the puzzle has five arms that we will call *flames*. There are two important principles to keep in mind when assembling your *Small Ball of Fire*. First, all pieces must have flames pointing in the same direction. In all the figures below the flames point clockwise as in figure 4.

Figure 4: Sculpture piece orientation

Figure 5: Assemblage of 2 pieces

Second, every two pieces of the puzzle will be attached in the same way. They will hook together at the top *and* the bottom, with a slot at the bottom of a flame sliding onto a slot on the top of another flame until the top of the flames are aligned. Between the flames that hook together there will be a flame on each piece, and these will slide past each other at the thinnest part of the flames (see figure 5). Every pair of pieces that are joined need to be joined in two places in this fashion. Never make one connection without making the other.

First we will assemble five pieces into what we will call a *ring*. This ring will correspond to one of the 12 pyramidal points of the small stellated dodecahedron. They will form the small pentagonal hole that you see facing you in the middle of the picture of the assembled sculpture in figure 1. Before we put these five pieces together we need to take a moment to think about the arrangement of the colored pieces.

This sculpture has been designed with the fewest colors needed so that two pieces of the same color never meet. This allows for a pleasing arrangement and makes the overall structure easier to discern.

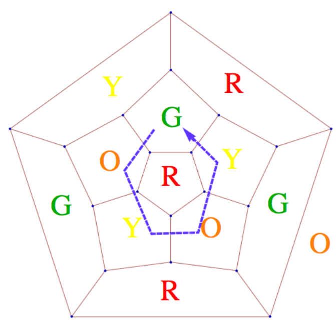


Figure 6: *A properly colored planar graph of the dodecahedron*

Figure 7: *Ring of five pieces*

A properly face colored planar graph of the dodecahedron provides a useful guide for properly coloring the sculpture (see figure 6). Begin with a green piece in your left hand and an orange piece in your right hand and attach them as in figure 5. Next move the orange piece to your left hand and pick up a yellow piece with your right hand and attached the orange piece to the yellow piece. Continue by adding another orange piece and then another yellow piece as in the configuration traced by the bent blue arrow in figure 6. Finally, attach the last yellow piece back to the green piece to complete the ring. It should resemble the photograph in figure 7. You should have a small pentagonal hole at the top and a wide opening at the bottom.

Figure 8: *The 6th piece in place (from below).*

Figure 9: *The first 6 pieces assembled (from above).*

Next insert a red piece into the middle of this ring. The easiest way to do this is to flip over the entire ring and insert the red piece from the underside. Since we are working on the "inside" we need the red flames to point counter-clockwise. To get the red piece in position, bend the five flames of the red piece and wiggle them through the five holes that surround the small pentagonal hole. It should look like figure 8 when the red piece is in place.

Flip the ring back over and attach each flame of the red piece to the two flames adjacent to that flame in the ring. You've now used half the pieces and the structure should start taking shape. You have a pentagonal ring above a horizontal red piece as shown in figure 9.

Rotate the puzzle so that the green piece is horizontal. This is the center of the next ring we will complete. This ring is marked on the planar graph in figure 10 by the bent blue arrow. We have already assembled the orange, red, yellow part of this ring. We just need to insert the yellow piece into a new red, and then this red into a new yellow, and then this yellow back into the existing orange. You will need to weave these pieces through the gaps before attaching the slots but this will make sense as you do it. Make sure none of your flames get stuck on the inside of the sculpture. You will notice many more places you can now attach these pieces to the existing structure. Again be sure you attach the top and the bottom of each pair of pieces you hook together.

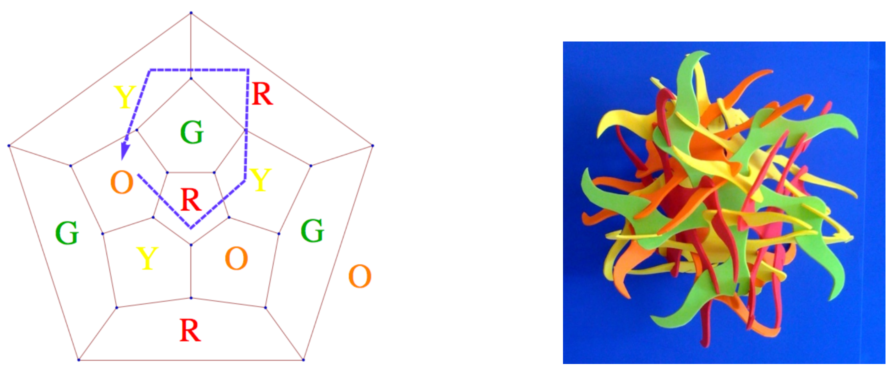


Figure 9: *The ring above the green piece*

We only have four pieces left to add. Pick another ring to complete. Carefully examine the planar graph and your sculpture so you are sure you know how they correspond. Suppose we choose to complete the ring above the right most yellow piece in the planar graph. We already have the red, green, red, and orange pieces attached above the horizontal yellow piece. Just add a green piece to complete this ring. Be careful to weave the flames through all the correct holes. Keep carefully following the diagram and completing rings until one piece is left.

Arrange the structure so that this last piece will be horizontal once inserted. You should have a complete ring with a pentagonal hole surrounded by 5 incomplete rings. You will need to undo the complete ring to insert the last piece, then reassemble that ring and attach the last piece to complete the other five rings.

5. Conclusions

Once the sculpture is assembled it is easy to see how it is derived from the small stellated dodecahedron. A properly face colored dodecahedron can be seen at the center and the open structure allows one to see how the faces of the dodecahedron intersect when extended. The proper coloring makes it easy to distinguish the faces.

This sculpture introduces concepts from solid geometry and graph theory in a fun, hands-on way. Although the assembly is very prescriptive, enough knowledge is gained for the builder to pursue a more thorough understanding of stellation based sculptures and eventually design her own.

References

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