An Exploration of Froebel's Gift Number 14 leads to Monolinear, Re-entrant, Dichromic Mono-Polyomino Weavings

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Abstract

When Froebel, the inventor of the Kindergarten [1] designed the "Gifts" and "Occupations" given to the children, he deliberately selected materials that provided a haptic dimension to their explorations. This physicality in the interaction with the gifts can create a significant potential learning focus making full use of concepts of spatial reasoning (front-back, over-under, etc.). For adults playing with the materials for the first time, and incorporating a reflective component in their doing and their thinking, the Gifts can provide a novel perspective on other, deeper mathematical concepts. The following paper and its accompanying workshop present some activities possible with Gift # 14, which involves the Occupation of paper weaving, and explore ideas in modular arithmetic, combinatorial geometry, ethnomathematics and more.

Keywords: Ethnomathematics, tessellations, dichromic tessellation, polyominos, monohedral tessellation, weaving, mat weaving, monolinearity, kindergarten, Froebel's Gifts, Sona sand drawings, counterchange symmetry

Introduction: Froebel's Gifts

In the early tradition of the Kindergarten Movement [1-4], children were given a succession of "Gifts" over time that were designed to develop their visual and spatial senses. Number 14 of these Gifts is variously identified as paper weaving [1, p. 78-83], paper interlacing (NEA, 1877, as cited in [4], p. 74) or mat weaving [3, throughout]. In essence, the Gift consists of a set of slit papers and strips of paper that can be woven through them. Figure 1 shows an example of work made using the gift.

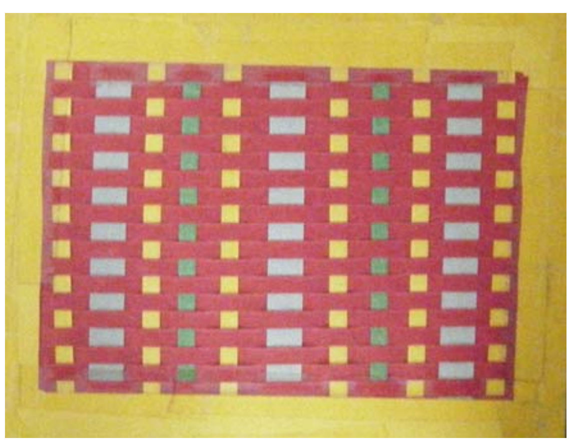

Figure 1: *Example of use of Froebel's Gift # 14 from one of the author's childhood*

Though the meaning and experience of "Kindergarten" has changed since, some of these materials are still available to children and others interested in crafts, predominantly in German-speaking countries. Figure 2 shows a contemporary sample of the slit paper.

Figure 2: *Currently available version of Froebel's Gift # 14*

The highly structured ways in which the materials of Gift #14 are meant to be used makes them a rich medium for some mathematical explorations related to weaving, including number work, geometry, and tessellations. In this paper and the connected workshop, readers and participants will have a chance to explore some of these mathematical ideas and their manifestation in concrete form.

Some Definitions

Mat Weaving: In the introduction, we explained that the materials constituting Gift #14 are known under various names. This stems from the fact that they were not originally designed in the English-speaking context, and the various terms result from to divergent translations. A search through relevant literature can dispel the confusion. The use of paper strips and slit paper is called mat weaving in [3]. In addition, "mat weaving" is also the expression used to describe the manufacture of mats using bamboo strips [5-8]. Finally, works of ethnomathematics (for example, [9]), refer to "plaited mats".

Emery [10] discusses the terminology used in the textile art world. In discussing weaving patterns, she asserts that:

> *"The elements of either or both [warp and weft[1]] can be paired, tripled, or used in larger groups–regularly or irregularly disposed. If identical [multiple-element] units are used in warp and weft the plain weave[2] structure is often called 'basket' or 'matt[3]' weave."*

This description gives a clue as to what might make a weave "mat" or "matt": the essential aspect is the flatness of the "unit" that is woven. Consequently, for this paper, *mat weaving* shall refer to: *weaving using a primary unit (which will be called a* strand*) that is essentially flat (with a cross section that is at least 2:1 in proportion), and whereby the properties of dimension, colour and texture of the resulting surface are therefore primarily determined by the properties of dimension, colour and texture of the*

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[1] These terms refer to the two main directions of the woven fiber.

[2] Defined later in this paper.

[3] Note that the description in [10] uses a different spelling that suggests the doubling of yarn, through a kind of visual onomatopoeia. In this paper, we will use 'mat' to show that the unit is single but wider than it is thick.

wider faces of the units. Figure 3 shows two examples of mat weaving: plain weave, which can be thought of as over-under-over[4], and twill.

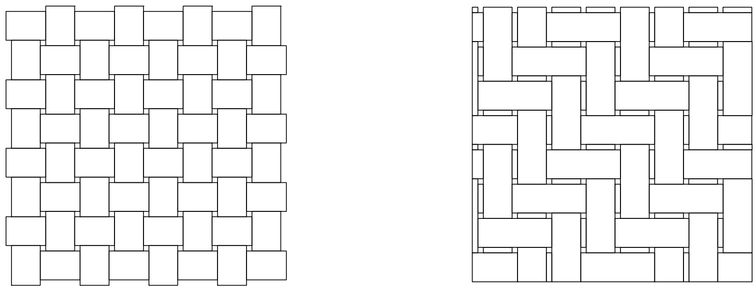


Figure 3: *Examples of patterns created from mat weaving: plain weave (left) and 2/2 twill*

When weaving with the contemporary Froebel's Gift materials, two mats with slits in perpendicular directions can be interwoven to create a single design of the same size. If two different-coloured pages are used, the design will be further emphasised. In essence, then, the colour can make explicit the direction of the visible element. Figure 4 shows the result of such a colouring for the case of the patterns of Figure 3.

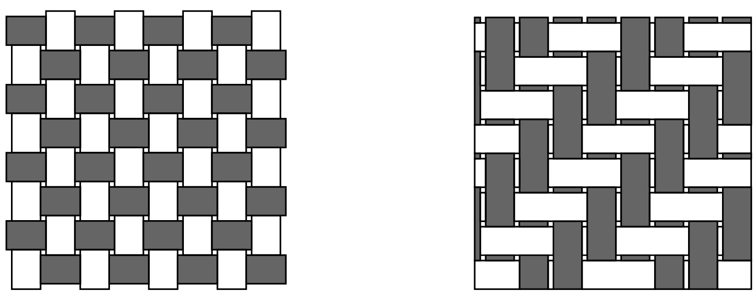


Figure 4: *Tessellations for the patterns of Figure 3, coloured according to strand direction*

Dichromic Tessellations: The result of colouring woven patterns according to the direction of the strands connects to research regarding the colouring of tessellations in general and the colour-preserving and colour-inverting symmetries of the resulting patterns. There have been several approaches and syntheses of findings [14-17]. In particular, [16, p. 64] discuss the idea of a "dichromatic" pattern as one whereby a rigid motion or symmetry transformation maps all the white (of one colour) regions onto black (of the other colour) regions, and vice versa. In the literature of art, this type of pattern is often said to incorporate a "counterchange". Figure 5 shows two examples, the standard chessboard pattern (which corresponds to the plain weave pattern above) and a pattern inspired from a Chinese lattice design found in [18, p. 235]: "Balanced Recurving Waves"[5].

The two examples shown in Figure 5 can be used to illustrate an additional property that the authors of [14-17] do not specify. In both cases, and if we consider that the Chinese lattice pattern is made of 3 different types of tiles, the colouring of the tiles is sufficient to specify their size and shape, without recourse to outlines, because each tile only shares edges with tiles of the 'other' colour. In contrast, the colouring of the twill weave in Figure 3 does not have this property, even though it is a dichromatic pattern according to the cited work, unless one considers infinitely long tiles. For the present paper, we define as *dichromic* those tessellations that have the additional property whereby it is not necessary to include outlines to define the individual, finite tiles.

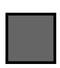
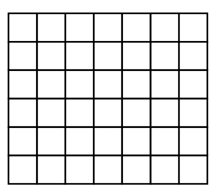
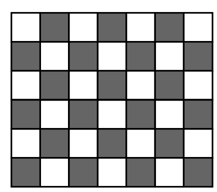
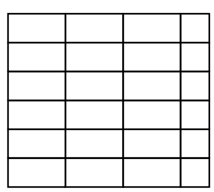
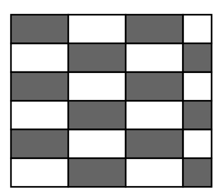
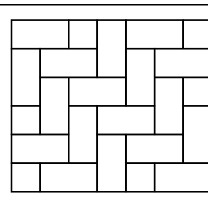
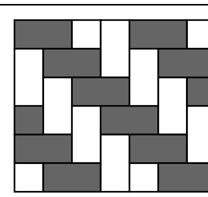
[4] This kind of pattern is generally assumed for Celtic interlace patterns in that it is not even discussed, but applied *en bloc* [11-13]. It is also the most stable.

[5] The mathematics behind whether a tiling can be coloured in this way is connected to graph theory and is exemplified in a problem posed in [19, p. 10]

Figure 5: *Examples of dichromic tessellation:*
chessboard tessellation (left) and "Balanced Recurving Waves"

Mono-polyomino tessellation: By definition and barring an unusual use of the slit paper, patterns created are based on the square grid and its various colourings with a maximum of two colours. In the case of the Chinese lattice pattern, the design contains three different tiles, each of which occurs in both colours: a 1×1 square, a 2×1 rectangle, and a 3-square L-shaped tile. These and other shapes defined as "shapes made by connecting certain numbers of equal-sized squares, each joined together with at least one other square along an edge" are collectively known as *polyominos* [20, p. 3]. Their names can be further specified according to the number of squares that are connected. For example, the Chinese lattice design of Figure 5 is made of 36 *monominos*, 48 *dominos* and 36 *trominos*. In addition, according to [14, p. 20], a tessellation is *monohedral* if "every tile in the tiling is congruent (directly or reflectively)". Together, these two definitions serve to specify that a *mono-polyomino tessellation* is a tessellation whereby the tiles are all congruent and, in addition, are polyominos. To illustrate: in Figure 4, the chessboard is a *dichromic mono(hedral)-monomino tessellation* and the Chinese lattice design is a *dichromic* (non-monohedral) *polyomino tessellation*.

Table 1 shows a few examples of polyominos and their monohedral tessellations. Note that all the illustrated dichromatic tessellations are dichromic according to the above definition with the exception of the twill pattern.

Designation	Basic tile	Example of tessellation	Dichromatic tessellation
Monomino (plain weave)			
Domino			
Domino (twill)			

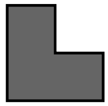
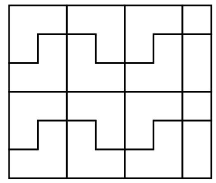
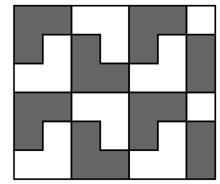
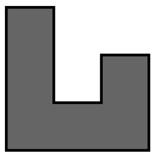
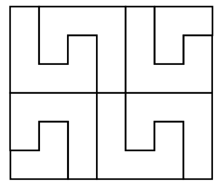
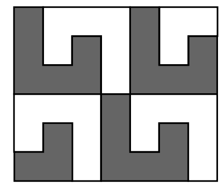
Tromino			
Hexomino			

Table 1: *Examples of polyominos and their tessellations*

Sona: Independent of the constraints of Froebel's Gift materials, the execution of a dichromic mono-polyomino design can be enhanced further by planning it in such a way that a single strand is used continuously throughout the entire design, alternately running in each direction, and re-entering at the edges. The technique required to achieve this is similar to that used in some of the documentation about mat weaving cited at the beginning of the article. In several cases, the weave lies diagonally to the mat's edges and the bamboo strips are folded at an angle so that they re-enter the mat in the other direction of weave, at 45° to the edge. Figure 6 shows a diagram of the re-entry for several adjacent strands.

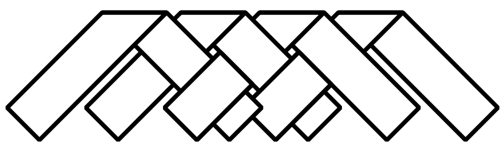


Figure 6: *Diagram showing re-entry of multiple adjacent strands*

In [9, p. 157-205], Gerdes explores a style of sand drawing that can contribute to the thinking involved in this design process. *Sona* were traditionally created in certain African tribes during story-telling, as a visual support. They frequently represented elements of the story, and were created by running a finger between previously marked grid points, as illustrated in Figure 7 (left). The design is constituted of a number of parallel lines connected at the edge in a "re-entry" similar to that in Figure 6. In addition, (right), the lines can be re-interpreted as bypassing each other "over" or "under" as in mat weaving.

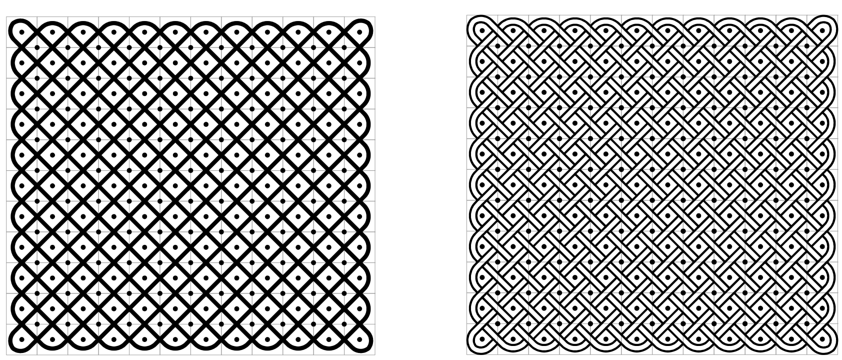


Figure 7: *Example of Sona: line drawing (left) and corresponding interlace pattern*

As previously noticed, the simplest and most stable design is achieved by using the "plain weave" structure, but it is by no means the only option. Figure 8 shows the same pattern using a wider strand. It results in the mono-monomino tessellation. The diagram on the right shows the colouring according to strand direction.

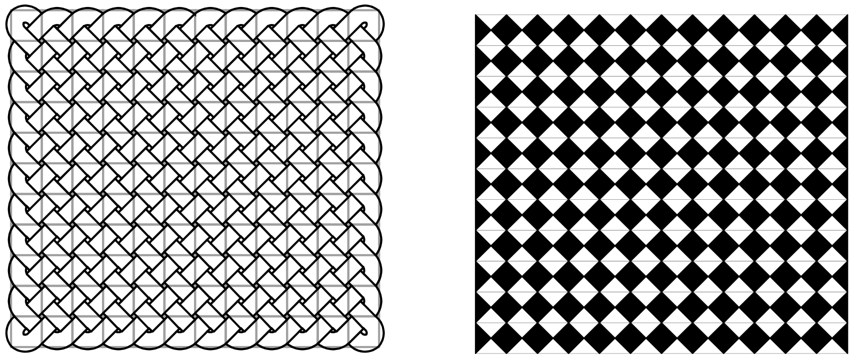


Figure 8: *Sona re-interpreted as mat weaving; coloured according to strand direction*

Through the use of 2-coloured ribbon, emphasis on strand direction can be preserved whilst creating the design with a single strand. This works because the re-entry technique involves a 45° fold of the ribbon, reversing its colour at the same time as it changes its direction by 90°, as shown in Figure 9.

Figure 9: *Re-entry of strand: application using 2-coloured ribbon*

Monolinearity: The concepts inherent to Sona [9] explain how only one strand might weave an entire surface. Figure 10 shows two completed Sona drawings in which multiple strands were needed. In both these cases, a single strand will not complete the pattern. The condition for what [9] calls *monolinearity* (i.e. "composed of only one line", (p. 161[6]) is that 1 be the highest common factor between the two dimensions, i.e. they be co-prime.

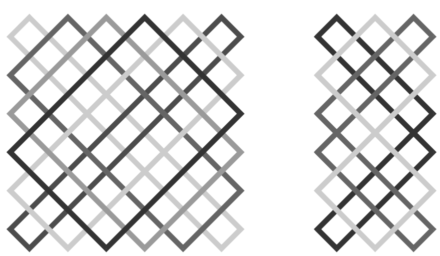


Figure 10: *Individual Strands in various dimensions of Sona mat weave:*
a 6×6 square requires 6 (left); a 3×6 rectangle requires 3 (right)

The designs of Figures 7 and 8 are monolinear because their dimensions are 11×12, where 11 and 12 are co-primes.

[6] See also [19, p. 48] for an analogous problem.

Synthesis

All the ideas discussed in the above sections can be incorporated into a single piece of weaving. Figure 11 shows a *dichromic mono-hexomino weave* where the colouring associated with the strand direction is used to create the individual tiles. Figure 12 shows the template for the same design, laid out to create a *monolinear, re-entrant* version of the tessellation (left), and the pattern in its 2-coloured ribbon version.

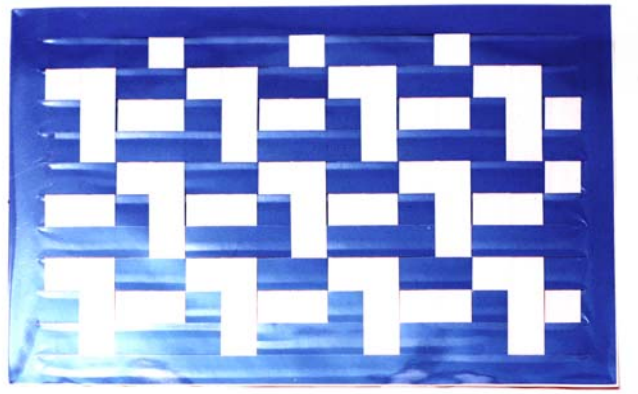


Figure 11: *Dichromic weave of mono-polyomino using Froebel's gift*

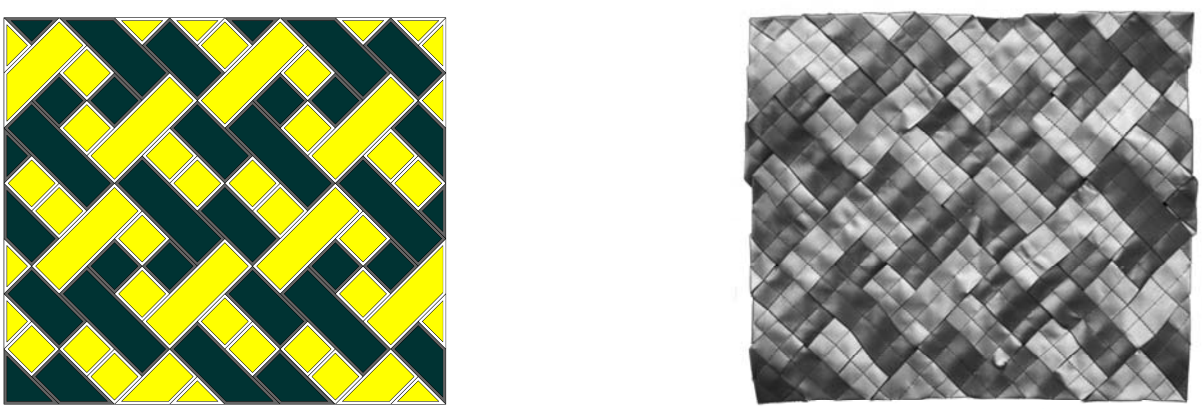


Figure 12: *Template (left) for and 2-coloured ribbon version (right) of a 7×8 monolinear, re-entrant, dichromic mono-hexomino weave*

Workshop Activity

In the workshop, participants will explore the processes described above, following a simplified, parallel process. In particular, there will be some time given to exploring mono-polyomino tessellations, selecting a polyomino and seeing if it can be tessellated into a dichromic regular pattern, then preparing a template and weaving the tessellation, using either Froebel's Gift #14, or 2-coloured ribbon.

Conclusion

Examples of educational applications of mathematics and art that are connected to symmetry often take the form of tessellation work, whereby children are asked to create Escher-like repeating patterns, using drawing, tracing, or sometimes, computer software. The freedom of these media in terms of possible forms makes the exercise one of reduction to what is possible in crystallographic terms. In contrast, with the use of a context such as mat weaving (as defined in the text), the constraints of the material and technique as well as the abstract mathematical ideas play into the design process. In fact, the interaction between the two sets of constraints (material and mathematical) creates a "third space" that can be educationally very rich and can appeal to multiple modes of learning.

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