Conformal Tiling on a Torus

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Abstract

Given a regular tiling of the torus, we want to depict it on a torus in space with as much conformal symmetry as possible. In particular, the conformal type of the surface should agree with that implied by the regular tiling, and symmetries not seen as Euclidean motions should be represented, whenever possible, by Möbius transformations.

There has been interest at Bridges [4] and elsewhere in finding symmetric embeddings of regular tilings on surfaces. A regular tiling on the torus is a quotient of one of the three regular tilings of $\mathbb{R}^2$ – by triangles, squares or hexagons – and most naturally the tiles should have this regular Euclidean geometry.

More generally, any doubly periodic pattern in the plane has translational symmetries by some lattice Λ. Up to similarity (rigid motion and scaling), we take the shortest nonzero vector in the lattice to be $(0,1)$; the second shortest (nonparallel) vector is then some (s,t) with $0 \le t \le 1/2$ and $s^2+t^2 \ge 1$. The quotient $T_{s,t} := \mathbb{R}^2/\Lambda$ is a flat torus, a parallelogram with opposite edges identified. The original pattern can be thought of as living on this torus.

Our goal is to draw a picture of such a regular tiling or other periodic pattern on a torus embedded in space. Of course a flat torus cannot be *isometrically* embedded in $\mathbb{R}^3$. But we suggest that nice pictures should at least use *conformally* correct tori. Recall that a map is conformal if it preserves all angles between curves. Of course similarities are conformal; other well-known examples include inversions in spheres (and hence all Möbius transformations), stereographic projection and the Mercator projection.

The conformal geometry of surfaces is well-understood mathematically via complex analysis. The Riemann mapping theorem says that any two simply connected domains in the plane are conformally equivalent; by extension any two topological spheres are conformally equivalent. (Any spherical tiling is most naturally drawn on the round sphere!) On the other hand, a thick annulus is not equivalent to a thin annulus; similarly not all tori are conformally equivalent. Instead, each torus is conformally equivalent to some flat torus $T_{s,t}$ as above; but these flat tori for different (s,t) are conformally distinct.

It is easy to show that any torus in space with mirror symmetry across some plane (or even inversion symmetry in some sphere) is conformally a rectangular torus (that is, a flat torus $T_{s,0}$ for some s). In particular, any torus of revolution is rectangular. Conversely, we can conformally embed any rectangular torus as a round torus via the map

$$T_{s,0} = \mathbb{R}^2/\Lambda \to \mathbb{R}^3, \qquad (x,y) \mapsto \frac{\left(s\cos {}^{2\pi x}/_{s},\, s\sin {}^{2\pi x}/_{s},\, \sin 2\pi y\right)}{\sqrt{s^2+1}-\cos 2\pi y},$$

which is obviously s-periodic in x and 1-periodic in y. Figures 1, 2, and 3 show regular tilings by 16 squares, 32 squares and 12 hexagons, respectively, embedded on the appropriate round tori via this map. This formula is not the "standard" parametrization of the round torus but instead is derived by embedding the flat torus isometrically in $\mathbb{S}^3 \subset \mathbb{R}^4$ and then using stereographic projection to map it conformally to $\mathbb{R}^3$. Note that the ratio of the major and minor radii of the image of $T_{s,0}$ is $R/r = \sqrt{s^2+1}$.

Many doubly periodic patterns with extra symmetry do fit exactly on a rectangular torus. In particular the five wallpaper groups $*2222$, $22*$, $22\times$, $**$ and $\times\times$ always imply a rectangular torus, while the three groups with four-fold symmetry (442, $*442$ and $4*2$) use in particular the square torus $T_{1,0}$.

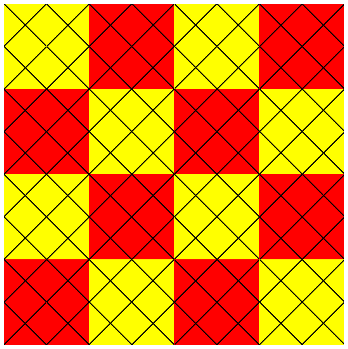
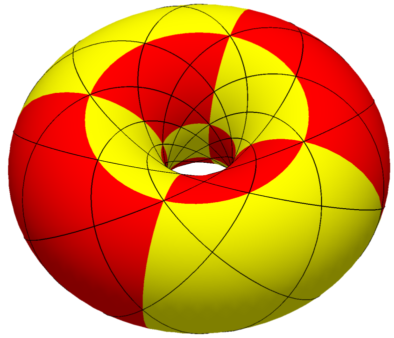

Figure 1: *A* 4×4 *array of squares (left) fits on the square torus* $T_{1,0}$*, which is conformally a quite thick round torus (right). The diagonal grid lines – always meeting at right angles – help to show the conformality. They form* $(1,\pm 1)$ *diagonals on the torus, each of which is a round (Villarceau) circle in space.*

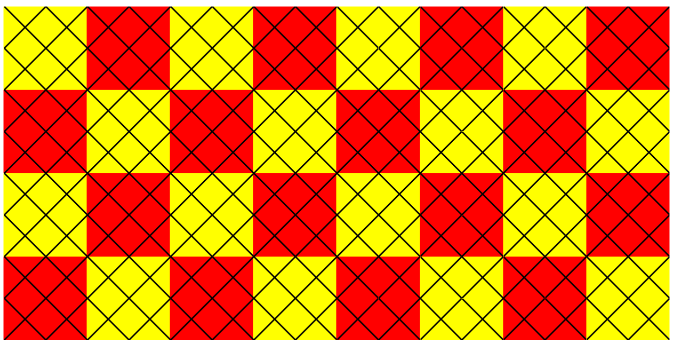
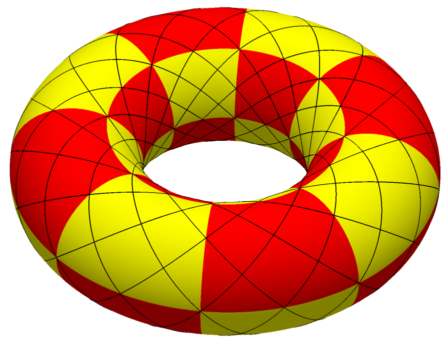

Figure 2: *An* 8×4 *array of squares (left) fits on the rectangular torus* $T_{2,0}$*, conformally a thinner round torus (right). The diagonal grid lines again meet at right angles, but are now* $(2,\pm 1)$ *diagonals on the torus.*

But other symmetric patterns fit most naturally on a rhombic torus. A lattice is *rhombic* if it is generated by two vectors of equal length. (In the coordinates above, we have $s = {}^1\!/_2$ or $s^2 + t^2 = 1$, depending on whether the rhombus has an angle smaller than $60°$). In particular, the symmetry groups $2*22$ and $*\times$ fit on any rhombic torus, while the five groups with three-fold symmetry (632, $*632$, 333, $*333$ and $3*3$) use in particular the hexagonal torus with $(s,t) = ({}^1\!/_2, {}^{\sqrt{3}}\!/_2)$.

As we have noted, a nonrectangular torus (in particular, a rhombic torus other than the square torus) is not conformally equivalent to any round torus or even to any torus embedded with mirror symmetry. To embed it conformally in space, we need to twist things in some way. One way to understand this intuitively is to note that the diagonals of the rhombus are unequal in length – thus the $(1,1)$ and $(1,-1)$ diagonals on the torus must have unequal lengths (in the appropriate conformal sense).

Ulrich Pinkall [3] has described a nice way to isometrically embed *any* flat torus into $\mathbb{S}^3 \subset \mathbb{R}^4$ as a *Hopf torus*, i.e., the lift (the preimage) of a closed curve $\gamma \subset \mathbb{S}^2$ under the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$. Indeed, any γ that has length $4\pi s$ and encloses a fraction t of the area of the sphere will lift to the torus $T_{s,t}$. Again, by stereographic projection, this isometric embedding in $\mathbb{S}^3$ yields a conformal embedding in $\mathbb{R}^3$.

On a Hopf torus, the $(1,1)$ curves are still round (Hopf) circles, but the $(1,-1)$ curves oscillate in the same way γ does, and are thus longer. Of course, given s and t, there is not a single natural choice for the curve γ. One idea is to minimize its elastic energy under the length and area constraints, perhaps also imposing certain symmetry. If we do this, the resulting tori are known to be constrained Willmore surfaces [1], that is, critical points for the Willmore bending energy given fixed conformal type. The surfaces shown in Figures 4 and 5 were generated this way, using Brakke's Evolver [2] to minimize the energy of γ.

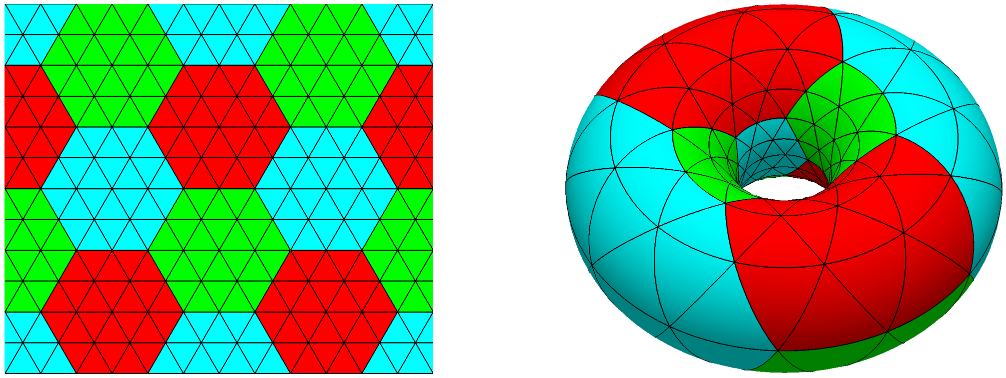

Figure 3: *This pattern of* 12 *hexagons (left) fits on the rectangular torus with aspect ratio* $s = 2/\sqrt{3}$, *conformally again a round torus (right). The grid lines shown, meet at* 60° *angles and form latitudes and* $(2,\pm 1)$ *curves on the torus.*

Probably the most famous regular tiling of the torus is that by seven hexagons; this map of seven countries cannot be colored with fewer than seven colors, since each pair of countries is adjacent. It has three-fold symmetry and thus lives on the hexagonal torus, which is (in a certain sense) the furthest from being rectangular. Any conformal embedding of the hexagonal torus will be strongly twisted, as in Figures 4 and 5.

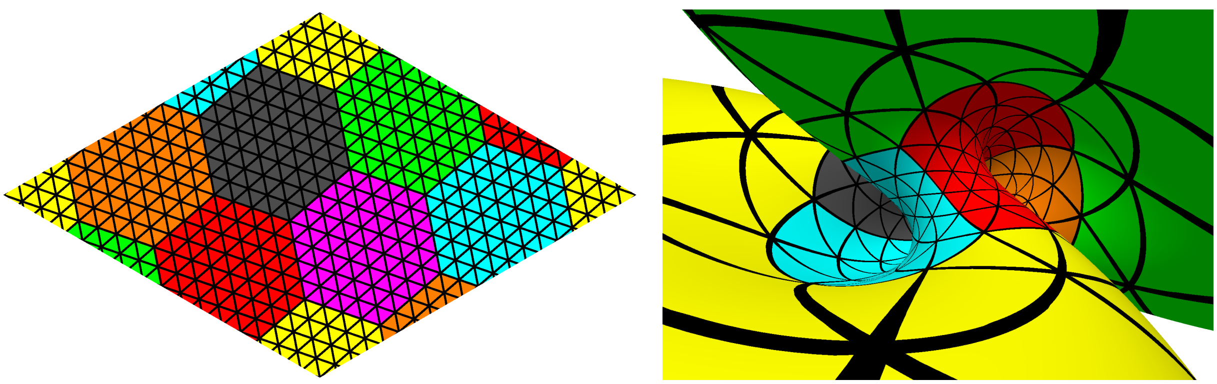

Figure 4: *The map of seven countries fits on the hexagonal torus, with a rhombic fundamental domain (left). If we conformally embed this torus, it will be highly twisted, far from having any mirror symmetry. A* 2*-fold Hopf torus, the lift of a curve* γ *looking like the seam of a baseball, has two huge spherical lobes (right). This figure and the next two use grids of constant-width lines in the flat metric; their width in the conformal pictures helps show the conformal stretch factor.*

Any flat torus has plenty of intrinsic symmetry: 2-fold rotations around any point and translations by any amount. (Rhombic and rectangular tori have in addition reflection symmetry, while the the square torus and the hexagonal torus even have higher-oder rotations.) If a rectangular torus $T_{s,0}$ is conformally embedded as above as a round torus in space, all its intrinsic symmetries are seen as Möbius transformations. (Indeed, before stereographic projection, they are seen as ambient isometries of $\mathbb{S}^3$.)

For nonrectangular tori, the situation is quite not as nice. Pinkall's Hopf tori have full translational symmetry in one direction (along the Hopf circles). Again, these symmetries are seen as rigid motions in $\mathbb{S}^3$ or Möbius transformations in $\mathbb{R}^3$. But the most we can hope for is discrete translational symmetry in the other direction, coming from symmetry of the curve γ. By choosing a 7-fold symmetric γ, for instance, we can arrange for a conformal picture of the map of seven countries (Figure 5, right) in which any country can be sent to any other by a Möbius transformation.

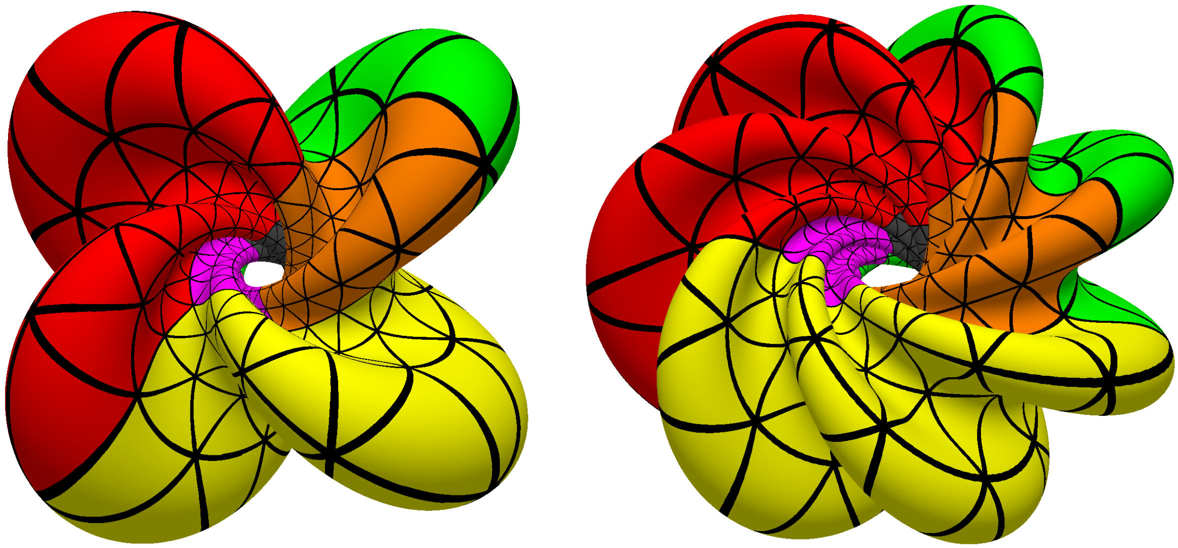

Figure 5: *The hexgonal torus can also be realized as a Hopf torus in other ways, for instance with 4-fold symmetry (left). For our regular tiling by seven hexagons, an especially nice version uses a 7-fold Hopf torus (right), where the seven countries are Möbius-equivalent to each other. The grid lines – consistent across Figures 4 and 5 – again demonstrate the conformality, since they clearly always meet at equal angles.*

Any rhombic torus is double-covered by a rectangular torus. (A rectangle cut along its diagonals reassembles to two rhombi.) Thus if we don't mind seeing two copies of our original pattern, we can embed it conformally on a round torus. For instance, Figure 6 shows a tiling by 14 hexagons, double-covering the map of seven countries, and fitting nicely on a round torus.

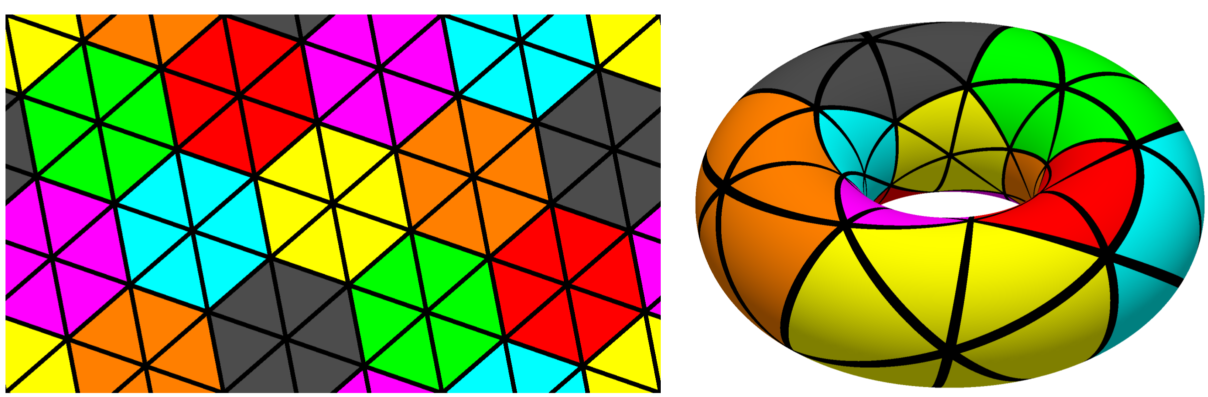

Figure 6: *This regular tiling of the torus by* 14 *hexagons also fits on the rectangular torus* $T_{\sqrt{3},0}$ *and thus conformally on a round torus. It is a double cover of the map of seven countries.*

References

[1] C. Bohle, P. Peters U. Pinkall, *Constrained Willmore surfaces*, Calc. Var. PDEs **32**, 2008, pp. 263–277.

[2] Ken Brakke, *Surface Evolver*, `www.susqu.edu/brakke/evolver/`.

[3] Ulrich Pinkall, *Hopf tori in* $\mathbb{S}^3$, Invent. math. **81**, 1985, pp. 379–386.

[4] Carlo Séquin, *My search for symmetrical embeddings of regular maps*, Bridges 2010, Pécs, pp. 13–24.