

## A Group Theory Approach to (re)Constructing Sol LeWitt's Drawing Series IV, #413

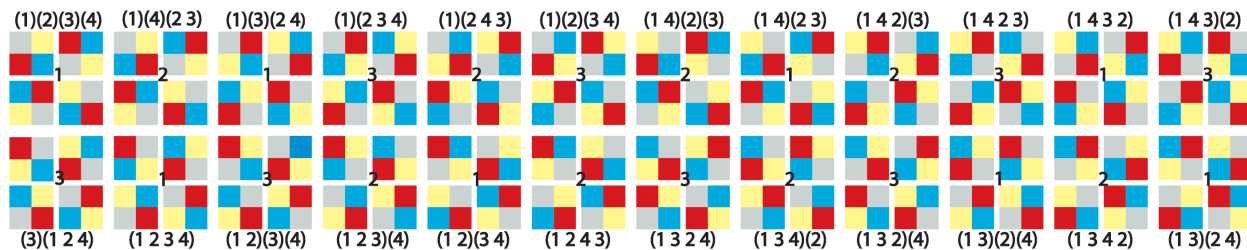
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### Abstract

The authors were inspired by *Drawing Series IV, #413*, to puzzle over what Sol LeWitt had in mind when he designed this wall drawing. The result is this paper, which describes a mathematical process involving group actions on a set by which LeWitt *might* have designed the piece. We also describe choice points in the process that are not mathematically determined and that produce interesting variations on a theme. The seed is one four-color square; the result is a beautiful and complex work of art.

### Introduction

Consider Sol LeWitt's, *Drawing Series IV, #413* [1] (referred to here as #413), represented in Figure 1. LeWitt was well known for his use of mathematical, particularly geometrical, ideas in constructing his works of art. See [2]. After several visits to the installation of #413 at the Massachusetts Museum of Contemporary Art (MASS MoCA), we were struck by the variety of mathematical ideas that seem to have been involved in the design of this wall drawing. There are many things that could be --- and have been --- noticed about the drawing, ranging from a multitude of small squares in four colors, larger squares containing four of the smaller squares each with a different color arrangement, and four of these four-color squares forming a larger unit that is separated by a wide white border. #413 gives us a visual and compelling model for puzzling over a structure within visually "irregular, colorful patterns." See [3]. We invite the reader to discover some of these interesting patterns. If you are reading this paper with diagrams in black and white, translate the four shades of gray, from lightest to darkest as yellow, gray, blue, and red. Figure 1 has references to permutations and orbits that will be discussed later in this paper.



**Figure 1:** *The layout of Sol LeWitt's Drawing Series IV, #413.*

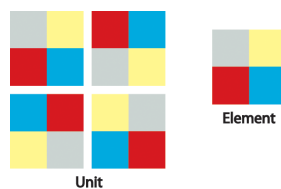
We noticed immediately on seeing #413 that there are 24 units set off by the larger white borders, and our mathematical imagination was drawn toward the idea of permutations. There is also a strong theme of symmetries of the square. For instance, within one unit, between the large white borders, each of the four four-color squares is a symmetry of the other three squares. Furthermore, LeWitt seemed to be

playing with factors of 24: Two rows of 12 units, four quadrants of six units, each with one of the four different colors in the upper left corner. But where were the factors three and eight? Motivated by our increasing number of observations, we sought to relate them within a mathematical structure that would explain all the parts of the drawing starting with one element, one seed, as it were.

### Framework for (re)Constructing #413

Here we describe how #413 could be constructed using a mathematical approach. We are not claiming that this was the method used by LeWitt, but that his result could be achieved this way. Whenever there is a choice point, that is, where the mathematics does not dictate the next step completely, we mark it with an \*. This is to allow us to refer easily to these choice points in the next section of the paper.

**Creating Elements.** An element\*, pictured in Figure 2, is the fundamental building block of #413. In trying to analyze #413, we initially thought of the four-color square as a planar object where the colors essentially number the vertices, and we began thinking of symmetries of a square. To some readers, it might seem more natural to think of the group of 24 permutations on four colors,  $S_4$ , as producing 24 unique four-color elements. However, our thinking about symmetries, while problematic at first, led us to a very powerful observation about the role in this wall drawing of orbits produced by a group action.



**Figure 2:** *Units and elements used to (re)construct #413.*

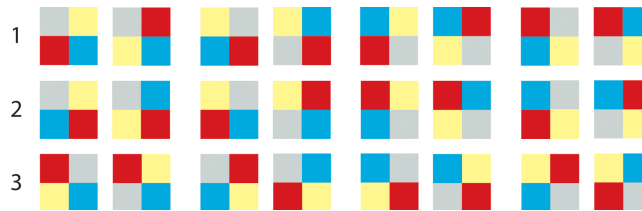
The problem comes with trying to reconcile the eight symmetries of a square,  $D_4$ , with the group of 24 permutations of four colors,  $S_4$ . Since a square is a rigid, planar object that can be rotated and flipped, but which cannot have one pair of adjacent vertices interchanged independent of the other pair of adjacent vertices, any permutation in  $S_4$  that would interchange only one pair of adjacent vertices of the square is not considered to be a symmetry of the square. But what if we thought of the square as a planar version of the Rubik's cube so that we could get from any vertex arrangement to any another? With paint (or ink wash in the case of #413), one can color the quadrants at will, thus making an impossible world (i.e., permutations that involve tearing apart and re-gluing) come to life.

By four-coloring the quadrants of a square and allowing them to interchange independently, LeWitt gave us a powerful visual way to consider the symmetries of the square in the context of permutations. Once we expand our thinking in this way, some interesting patterns emerge. One thing we notice is that only twice do we need to interchange just one pair of adjacent vertices in order to represent all the elements of  $S_4$  with four-colored squares. The squares in Figure 3 are not symmetries of each other; they represent the needed two interchanges that are not possible when thinking of the square as a rigid planar object. All 24 permutations can be obtained by the group action of  $D_4$  on these three elements.



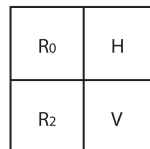
**Figure 3:** *Generating elements for each orbit.*

These three elements are representatives of the three equivalence classes of four-color squares, and each equivalence class is the orbit of one of the three squares under the group action of  $D_4$  on a square. We denote  $D_4$  by  $\{R_0, R_1, R_2, R_3, V, H, D_1, D_2\}$ , where  $R_0, R_1, R_2,$  and  $R_3$  are the  $0^\circ, 90^\circ, 180^\circ,$  and  $270^\circ$  rotations in a counterclockwise direction, and  $V, H, D_1,$  and  $D_2$  are the vertical, horizontal, main diagonal, and minor diagonal reflections, respectively. The 24 uniquely colored elements, classified into orbits, are shown in Figure 4. As shown, there are eight elements in each class, and each element is a unique symmetry of a square within that orbit. Now we have solved the mystery of where three and eight as factors of 24 come into play!



**Figure 4:** *Orbits of generating squares, in the order:  $R_0, D_1, V, R_1, D_2, R_2, R_3, H.$*

**Creating Units.** Each of the 24 unique elements, created above and pictured in Figure 4 (also referenced in cycle notation in Figure 1), is used once and only once to form a unique unit; one unit is pictured in Figure 2. Units consist of four four-color squares, laid out as follows: First, one of the 24 unique elements, referred to as the  $R_0$  element, is placed in the upper left quadrant of the layout shown in Figure 5. Then three symmetries of  $R_0$ , namely  $R_2, V,$  and  $H,$  are placed in the other three quadrants as indicated. In this way each unit is created by a subgroup,\* specifically the subgroup  $\{R_0, R_2, V, H\}$ , of  $D_4$  acting on the unique  $R_0$  element in that unit. We refer to the color of a unit by the color of the upper left corner of its  $R_0$  element. For example the unit in Figure 2 is called “gray.”



**Figure 5:** *Placement of elements within a unit.*

**Placing Units in the Final Layout.** Now we are ready to place\* the units. Start by dividing the installation (24 units in two rows of 12 units) into quadrants of six units each....but which six units go into each quadrant? Noting that there are exactly six units that are colored gray in the uppermost left corner, six that are colored yellow, six blue, and six red, we see that LeWitt separated the units by upper left corner color into quadrants.

Now we see where orbits come into play! The numbers in the middle of the units in Figure 1 refer to the orbits from which the units were chosen. To place units\* in the upper left quadrant, we choose the first gray unit appearing in orbit 1, then the first gray unit appearing in orbit 2, the second gray unit in orbit 1, the first gray unit in orbit 3, the second gray unit in orbit 2, and finally the second gray unit in orbit 3. Units are placed into the other quadrants in a similar manner, with the lower left quadrant having all red units, the lower right quadrant having all blue units, and the upper right quadrant having all yellow units. As a consequence of the placement method, no unit is placed next to or above/below a unit from the same orbit (row). Thus we see how LeWitt might have been playing with factors of 24: There are 2 rows of 12 units, 4 quadrants of 6 units, and 3 orbits of 8 units.

Within the above framework for (re)constructing #413, there are many more subtle patterns that can be noticed. For instance, the choice of orbits\* for placement of units follows a pattern that could be made by listing twice all of the ordered pairs (of non-repeating numbers) that can be made from the set {1, 2, 3}. Reading across the first row and then the second row, both left to right, we see the following pairs of orbits: (1,2), (1,3), (2,3), (2,1), (2,3), (1,3), (3,1), (3,2), (1,2), (3,2), (3,1), (2,1). Also notice that the order of orbits in the bottom row is exactly the reverse of the order of orbits in the top row. Another subtlety is that the arrangement of colors by quadrant is the same as the arrangement of colors for the very first square created, shown in Figure 2, and placed in the upper left corner of the upper left unit. Of course, there were also many artistic considerations guiding LeWitt as he constructed #413.

### **\*Variations on a Theme**

**Freedom Within Structure.** Within the above framework, at each of the marked choice points, there are decisions that can be varied, producing different units and different arrangements of units, which will affect the color array, adjacencies of colors, and the overall look of the finished piece. Some points where the process is not mathematically determined are: 1. the arrangement of the colors of the initial element, 2. the order in which the symmetries of  $D_4$  appear in the orbits, 3. the choice of adjacent vertices to interchange in constructing the orbits, 4. the subgroup that is used to create the units from the 24 elements, and 5. the placement of units into the finished piece. For example, in #413, there is maximum variability in the appearance of each color in the quadrants of the elements within a unit. If the subgroup  $\{R_0, R_2, D_1, D_2\}$ , is used to create the units, the colors do not appear in all quadrants, but instead each color appears twice in two quadrants. This alternative arrangement reduces the overall appearance of variability.

**A Piece of Art Inspired by #413.** Inspired by #413 and using folded paper squares (origami) that connect in groups of four to make squares that allow for interchange of vertices independently, we created a work for the Bridges 2011 art exhibit. We used the process outlined above, but created our units using the subgroup  $\{R_0, R_1, R_2, R_3\}$ . The outcome shows subtle differences that are possible within the framework. Because origami artists strive to create works with no tape or glue, and held together with only tab and pocket construction, we wanted our entire piece to hold together using only this technique. Thus we could not create obvious spaces between units in order to define them. Instead, we inserted color foil into the spaces in the middle of the four squares in a unit in order to define them as a unit and to denote their orbit origination. Due to the way that the our origami squares interlock, they are displayed “on point” rather than with edges adjacent, creating a different look and requiring a bit more work to interpret the process. The result, though inspired by #413, is very different and illustrates the richness of this exploration.

### **References**

[1] Sol LeWitt (American, 1928-2007), *Wall Drawing 413*, from *Drawing Series IV*, Ink and color washes, March 1984. Installation at the Massachusetts Museum of Contemporary Art, North Adams, Massachusetts, 2008-2033.

[2] Susan Cross and Denise Markonish (Eds.), *Sol LeWitt 100 Views*, MASSMoCA, North Adams, Massachusetts USA, 2009.

[3] Massachusetts Museum of Contemporary Art, *Wall Drawing 413*, <<http://www.massmoca.org/lewitt/walldrawing.php?id=413>>, accessed March 2011.