The Sacred Cut

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Abstract

A new project oriented course on the Mathematics of Design has been developed for NJIT's College of Design. A module from this course on the Sacred Cut is described.

Introduction

I have created a new course in Mathematics of Design for students from NJIT's College of Design. The course is project oriented, and there are no examinations. The students are graded on their designs, scrapbooks, homework, and essays. The topics covered in the course are: informal geometry, projective geometry constructions, theories of proportion, tilings and patterns, symmetry, fractals, graph theory applied to creating floorplans, knot theory and design, mirror curves and Lunda designs, three-dimension geometry and polyheda. Modules for each of these topics are posted on my website [1], and my book, Connections [2] is used as a text with Beyond Measure [3] as a reference. I have been collaborating in this endeavor with Slavik Jablan who teaches a related course at University of Belgrade [4]. Through each of these topics we show how mathematics informs design. I will relate one of the modules for the course based on the sacred cut.

The Sacred Cut

The golden mean and the golden section is well known to designers and mathematicians. Yet another number and proportion known as the silver mean and the sacred cut also holds great interest. The golden mean, $\phi = \frac{1+\sqrt{5}}{2}$, solves the equation: $x - \frac{1}{x} = 1$ or,

$$x^2 = 1 + x \tag{1}$$

while the silver mean, $\theta = 1+\sqrt{2}$, solves the equation: $x - \frac{1}{x} = 2$ or,

$$x^2 = 1 + 2x \tag{2}$$

These lead to the continued fraction expansions:

$$\phi = 1+1/1+1/1+1/\ldots \quad \text{and} \quad \theta = 2+1/2+1/2+1/\ldots \tag{3}$$

Eq. 1 leads in the geometric Sequence 4a to the phi-sequence and another sequence in which each term is doubled:

Blue	|	$\ldots \frac{2}{\phi}$	2	2ϕ	$2\phi^2$	$2\phi^3$		(4b)
Red	| $\ldots$	1	ϕ	ϕ^2	ϕ^3	$\phi^4 \ldots$		(4a)

These are the proportions of the Red and Blue sequences that make up the architect LeCorbusier's system of proportions called the Modulor [1]. These are both Fibonacci sequences with the property: $a_n = a_{n-1} + a_{n-2}$ and with corresponding integer Fibonacci sequences,

Blue sequence:	2 4 6 10 16 26 …	(5b)
Red sequence:	1 2 3 5 8 13 21…	(5a)

Notice that these sequences are approximate geometric sequences, i.e., $a_n^2 = a_{n-1}a_n \pm 1$ or $a_n^2 = a_{n-1}a_n \pm 4$. Each term of Sequence 5a is the arithmetic mean of a pair the terms from 5b that brace it while each term of 5b is the approximate harmonic mean of the pair of terms from 5a that brace it with the approximation getting better the further out in the sequence one goes (note: the harmonic mean c of a and b is c = 2ab/(a+b)) These relationships are exact for Sequences 4a and 4b. Also, the ratio of successive terms approaches ϕ in the sense of a limit. Whenever you find an additive property within the integer sequences that holds for all terms of the sequence, the same property holds for the ϕ- sequences. For example, 2+3+5=10 or 5+ 8+13=26, etc. so that $1+\phi+\phi^2 = 2\phi^2$, etc.

As a result of Eq. 2, the silver mean leads to another pair of geometric sequences, the θ-sequences,

$$\ldots \frac{\sqrt{2}}{\theta} \quad \sqrt{2} \quad \sqrt{2}\theta \quad \sqrt{2}\theta^2 \quad \sqrt{2}\theta^3 \quad \sqrt{2}\theta^4 \ldots \tag{6b}$$

$$\ldots \frac{1}{\theta} \quad 1 \quad \theta \quad \theta^2 \quad \theta^3 \quad \theta^4 \ \ldots \tag{6a}$$

with the property that $a_n = 2a_{n-1} + a_{n-2}$. Sequences with this property are called Pell sequences. Two such integer Pell sequences are:

1 3 7 17 41 99 239 577 …	(7b)
1 2 5 12 29 70 169 408 …	(7a)

Again the ratio of successive terms of each Pell sequence approaches θ in the sense of a limit. We also find that the ratios between successive terms of Sequences 7b and 7a approach $\sqrt{2}$ in the sense of a limit. Also notice that these sequences are again approximately geometric and that each term of Sequence 7a is the arithmetic mean of a pair of terms of 7b while each term of the Sequence 7b is the approximate harmonic mean of a pair of terms from 7a with the approximation getting better the further out in the sequence. Again, these relationships are exact for Sequences 6a and 6b and contain many additive properties, e.g.,

2 + 3 + 7 + 5 = 17 or 5 + 7 + 17 + 12 = 41, etc.

We find that these additive properties are again inherited by the pair of θ sequences. In fact, if the terms of 7a are doubled, it results in a third sequence which guarantees yet more additive properties. These Pell sequences proliferate into an infinite sequence of Pell sequences guaranteeing yet more additive properties [2].

Geometry of the Sacred Cut

Both ϕ and θ series are of interest to designers because of their many additive properties. Let us now consider the geometry of the sacred cut [2]. In Fig. 1 an arc of a circle is drawn with compass point at a vertex of a square through the center of the square and divides the side in the sacred cut ratio θ:1. Four such arcs initiated from the four vertices of a square in Fig. 2 creates a regular octagon, and they divide the square into three kinds of rectangles with proportion: 1:1, a square (S); 1: $\sqrt{2}$, a square root rectangle (SR); and 1: θ that I refer to as a Roman rectangle (RR) shown in Fig. 3. Such a subdivision was discovered by Donald and Carole Watts on a tapestry in one of the Garden Houses of Ostia, the port city of the Roman Empire [5] (see Fig. 4).

The additive properties of the θ-series are reflected in the manner in which these three species of rectangles are related to each other. For example, Fig 5 shows that when S is added or subtracted from an SR it results in a RR so that if a double square DS is added to an RR it results in a RR at a larger scale. Also, a pair of SR's results in an SR at a larger scale whereas cutting an SR in half results in a pair of SR's at a smaller scale, i.e.,

$$\text{SR} \pm \text{S} = \text{RR}, \ \text{S} + \text{S} + \text{RR} = \text{RR}, \ \text{and} \ \text{SR} + \text{SR} = \text{SR} \qquad (8)$$

This enables one to begin with an S, and with compass and straightedge create an SR. Eqs. 8 can then be used to create S, SR, and RRs at different scales. This leads to a project in which the student creates a tiling with S, SR, and RRs at three different scales. One such tiling created by Marc Bak, a student in my class, is shown in Fig. 6.

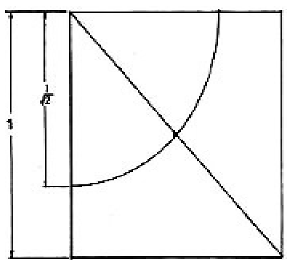


Fig. 1. Sacred Cut. The side Is divided in the ratio 1: θ .

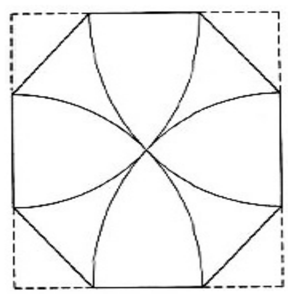


Fig. 2. Construction of a regular octagon

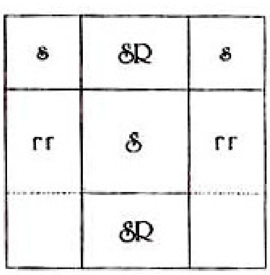


Figure 3. *Subdivision of a square*

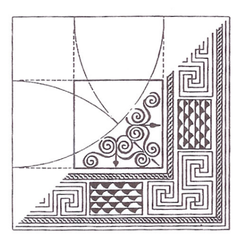
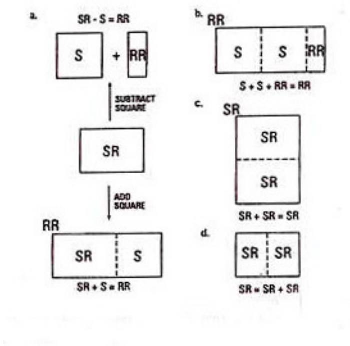


Figure 7.5 Mosaics and paintings in the Garden Houses of Ostia are in many cases laid out according to the geometry of the sacred cut. This photograph, following the pattern of Figure 7.4, shows a floor mosaic found in one of the houses. (By Tom Prentiss, photographed by John Moss. Reprinted with permission by Scientific American.)

Figure 4. *A tapestry in a Garden House of Ostia*



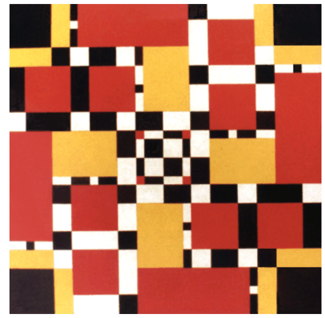


Figure 5. *Relationships between S, SR, and RR* **Figure 6**. *Sacred Cut tiling by Marc Bak*

Margit Echols Quilt

Margit Echols, a geometric quilter, connected various boundary points in Fig. 2 to create the five patterns shown in Fig. 7. She then combined these patterns into the quilt shown in Fig. 8

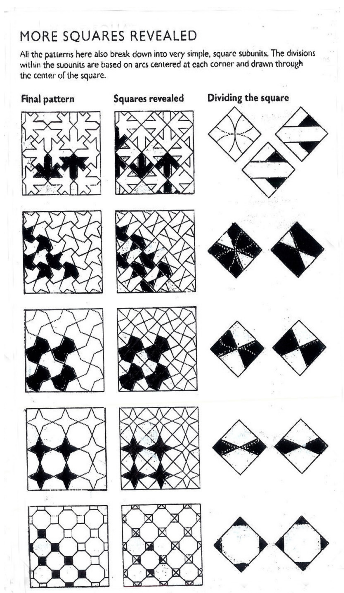


Figure 7. *Sacred cut patterns in a square.*

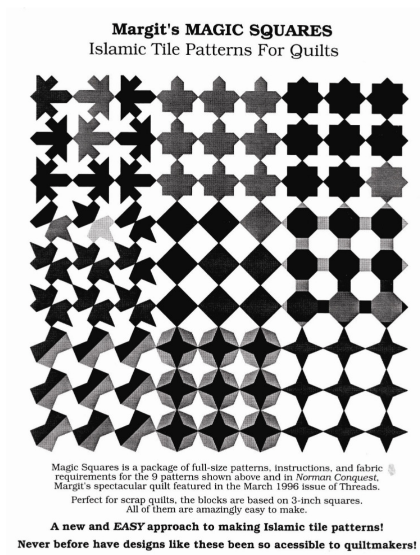


Figure 8. *Margit Echols' Quilt*

Conclusion

The sacred cut illustrates how algebra, number, and geometry are woven together to create designs. This is the spirit I wish to convey in my new Mathematics of Design course.

References

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[2] J. Kappraff, *Connections,* Singapore: World Scientific, 2001
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[4] S. Jablan, G8G4 Compact Disc for the Gathering for Gardner conference.
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