Pavimenta: A Photo Tiling Tool

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Abstract

It is well known that there are seventeen wallpaper groups, that generate distinct tilings of the plane. Some of the famous Escher drawings are based on these groups. The *Pavimenta* program [2] uses these seventeen groups to generate tilings of the plane using fundamental regions taken out of any image. Its ease of use makes it straightforward to create surprisingly beautiful tilings.

1 The Seventeen Wallpaper Groups

Plane tilings have been used thoughout the centuries to create artistic geometric figures. All seventeen plane isometry groups have appeared in ornamental creations, although only in the late 19th century Fedorov and Schoenflies have proved that there are only seventeen of these groups. A simple proof comes from the fact that the quotient of the plane by a plane isometry group is an euclidean 2-orbifold, which has null Euler characteristic. The Euler characteristic $\chi(\mathscr{O})$ of an orbifold with underlying surface S, k cone points of orders $n_1, \ldots, n_k$ and t corner points of orders $m_1, \ldots, m_t$ is

$$\chi(\mathscr{O}) = \chi(S) - \sum_{i=1}^{k} \left(1 - \frac{1}{n_i} \right) - \frac{1}{2} \sum_{j=1}^{t} \left(1 - \frac{1}{m_j} \right).$$

Since the only surfaces with nonnegative Euler characteristic are the sphere (with $\chi(S) = 2$), the projective plane and the disk (with $\chi(P) = \chi(D) = 1$), the torus, the annulus, the Klein bottle and the Möbius band (with $\chi(T) = \chi(A) = \chi(K) = \chi(M) = 0$), a straightforward calculation shows that there are only 17 possibilities for $\{S, \{n_i\}, \{m_j\}\}$.

These plane symmetry groups have been described by different notations, some of which are given in the table below. The Montesinos notation [1], indicates the underlying surface of the quotient orbifold, the orders of the cone points, and the orders on corner points, in the form $Sn_1 \ldots n_k(m_1 \ldots m_t)$.

2 The *Pavimenta* Program

There are several programs that generate tilings of the plane using the seventeen plane isometry groups. It is possible, using these programs, to make a drawing on the plane and see automatically how it is reproduced by the isometry group selected. The *Pavimenta* program, besides being capable of doing this, lets the user choose any image and see the effect of the isometry group on it. The interface is simple, with three areas on the screen.

The upper right area has seventeen buttons corresponding to the plane isometry groups. Clicking each one of these buttons changes the image on the main region. The lower right area shows the fundamental region associated with the selected isometry group. This fundamental region is either a triangle or a quadrilateral, according to the isometry group selected. Since the image chosen by the user is a rectangle, the

Group	Montesinos	Conway-Thurston	Crystallographic	maximal order of rotation	mirrors
1	T	o	p1	1	No
2	A	**	pm	1	Yes
3	K	oo	pg	1	No
4	M	*o	cm	1	Yes
5	S2222	2222	p2	2	No
6	P22	22o	pgg	2	No
7	D(2222)	*2222	pmm	2	Yes
8	D22	22*	pmg	2	Yes
9	D2(22)	2*22	cmm	2	Yes
10	S442	442	p4	4	No
11	D(442)	*442	p4m	4	Yes
12	D4(2)	4*2	p4g	4	Yes
13	S333	333	p3	3	No
14	D(333)	*333	p3m1	3	Yes
15	D3(3)	3*3	p31m	3	Yes
16	S632	632	p6	6	No
17	D(632)	*632	p6m	6	Yes

Figure 1 : *The seventeen plane symmetry groups.*

fundamental region contains only a subset of this image. The user can draw in this fundamental region, whether it contains an image or is blank, and see immediately the result of the drawing on the main area. Right clicking on the fundamental region lets the user select another image.
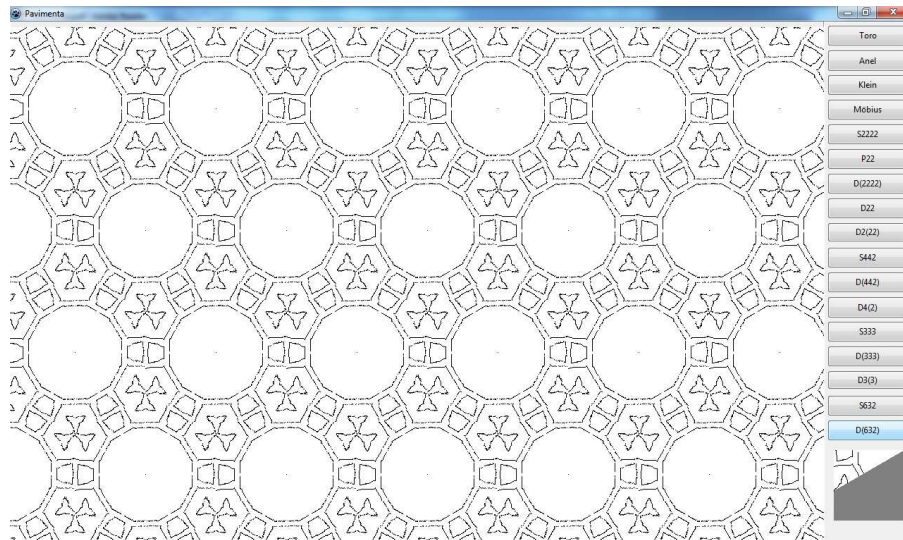

Figure 2 : *The* Pavimenta *interface.*

The main area shows the image generated by the selected isometry group on the current image. The program uses standard linear algebra to rotate, translate and reflect the fundamental region to the various

positions on the screen. These calculations are done for every change made on the fundamental region, in real time.

Although it would be possible to construct these images with a standard photo editing program which is capable of applying linear transformations to pictures, *Pavimenta* generates them with a single click, which makes it extremely easy to create beautiful and surprising images. Below are some examples of tilings produced with *Pavimenta* and the images that originated them.
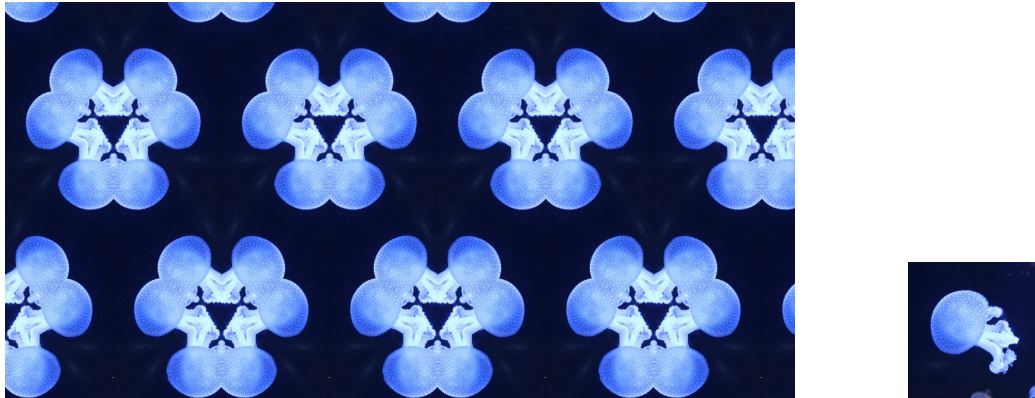

Figure 3 : *A tiling of an anemona by D(333).*

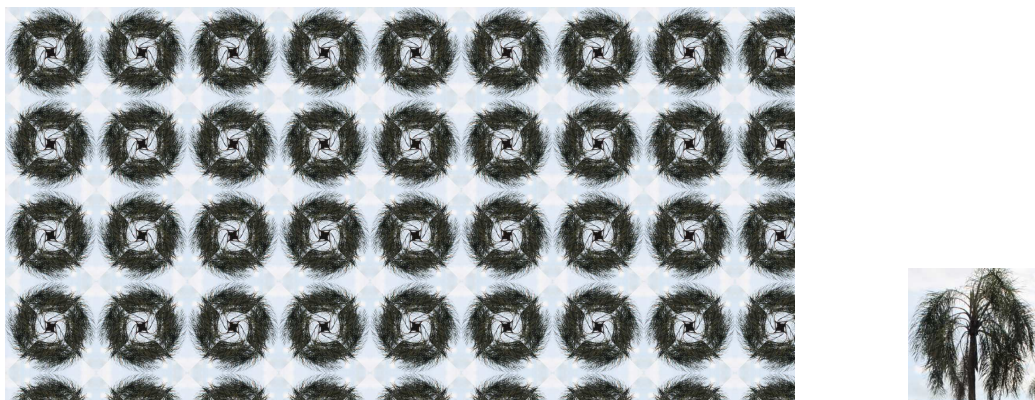**Figure 4** : *A tiling of marbles by D(2222).*

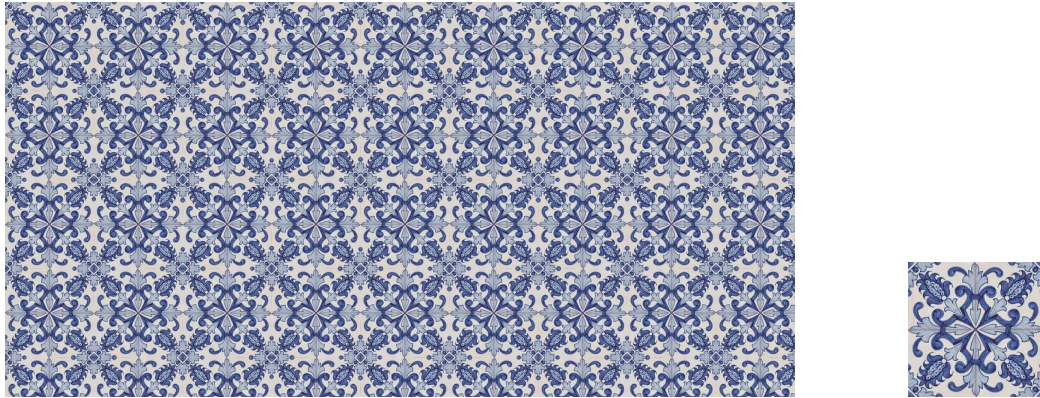**Figure 5** : *A tiling of palm trees by D4(2).*

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Figure 6 : *Since this portuguese* azulejo *is already highly symmetrical, all tilings from T to D4(2) produce the same image.*
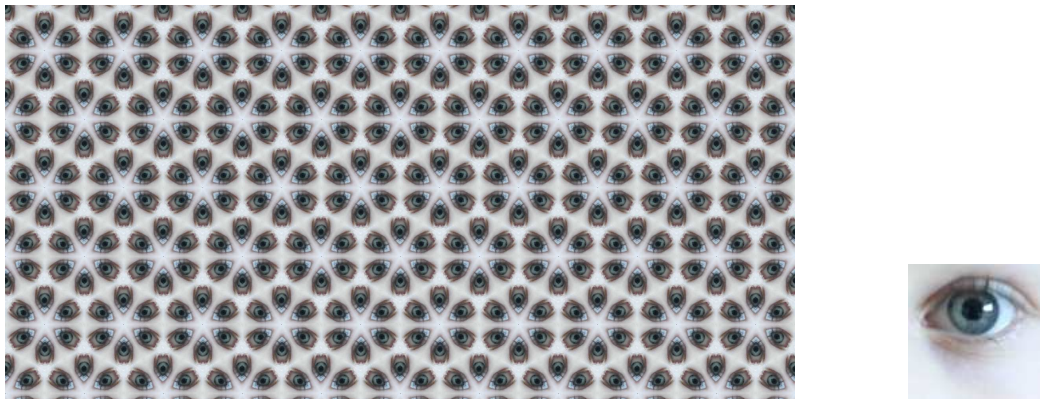

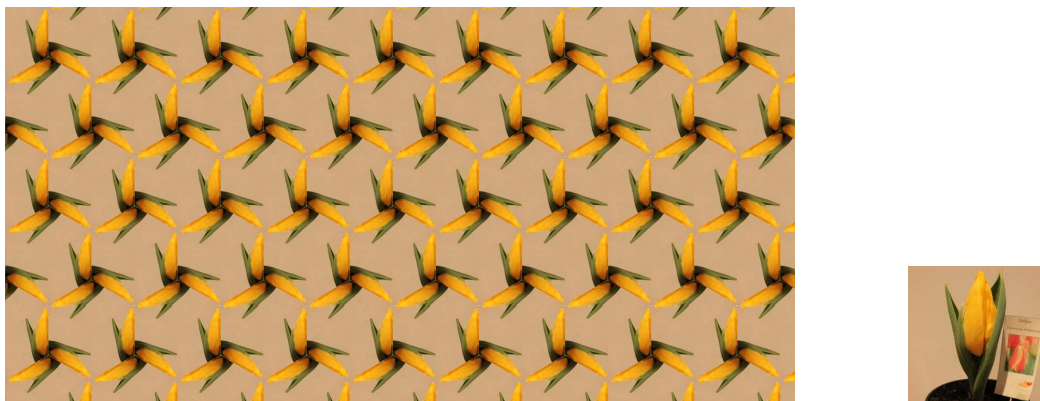**Figure 7** : *A tiling of eyes by D(632).*

Figure 8 : *A tiling of tulips by S(333).*

References

[1] Montesinos, J. M., Classical Tesselations and Three Manifolds, Universitext, Springer, Berlin (1987).

[2] Salgueiro, A., *Pavimenta*, executable available at http://www.mat.uc.pt/∼ams/Pavimenta, (2011).