Bending Hyperbolic Kaleidoscopes

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Abstract

We demonstrate different ways to create and visualize tilings made with hyperbolic kaleidoscopes. We start with tiling of the hyperbolic plane and build bended kaleidoscopes in the hyperbolic space.

One can build only 4 different kinds of kaleidoscopes in two dimensional euclidean plane. In orbifold notations[1] these kaleidoscopes are *422, *333, *632, *2222. The hyperbolic plane admits infinite number of different types of kaleidoscopes, most of which also have independent continuous parameters defining the shape. Hyperbolic plane kaleidoscopes were first used in art by M.C.Escher in his four *Circle Limit* woodcuts. All M.C.Escher's prints and most of other existing hyperbolic art work (see [2,3] for example), with rare exceptions[1,4] are based on the triangular hyperbolic kaleidoscopes. Three angles at the corners of triangular hyperbolic kaleidoscope should be sub-multiples of π (π/n, $n > 1$) or zero and the sum of the angles has to be strictly less than π. These conditions are satisfied by infinite number of hyperbolic triangles. Hyperbolic triangles are rigid, their shape and size is fixed by fixing the angles. Fig 1. shows examples of tiling formed by triangular hyperbolic kaleidoscopes presented in the Poincare disk model.

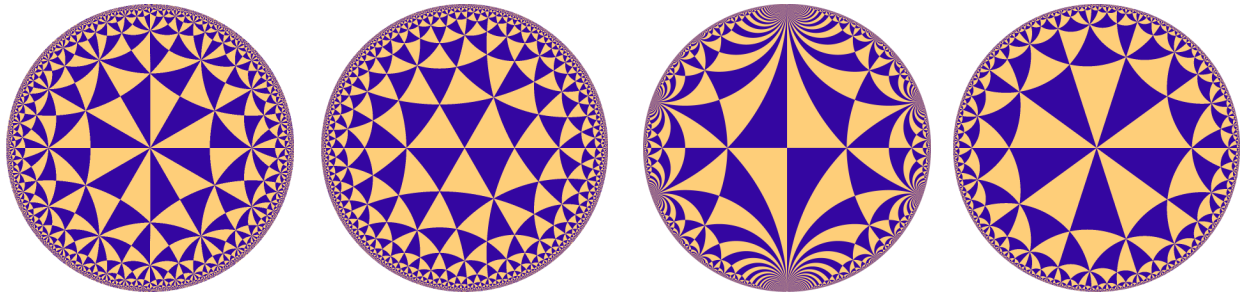

Figure 1: *Triangular hyperbolic Kaleidoscopes *642, *334, *24∞, *543*

Hyperbolic geometry admits much wider set of kaleidoscopes beyond triangles. In general, an arbitrary polygon with arbitrary number of sides and angles, which are sub-multiples of π can be used to make hyperbolic kaleidoscope. Fig.2 shows tiling generated by quadrilateral, pentagonal, hexagonal and and octagonal kaleidoscopes. The shape of the hyperbolic *n*-gon has *n-3* degrees of freedom: lengths of some *n-3* sides. Euclidean plane has such similar flexible kaleidoscope *2222 - rectangle with arbitrary aspect ratio. Fig 3. shows few tiling made by pentagonal hyperbolic kaleidoscopes with all right angles *22222. Lengths of two sides of pentagon are free parameters.

We can get much wider variety of kaleidoscopes by bending the fundamental polygon in three dimensions. The simplest hyperbolic polygon which can be bended is quadrilateral *3222 (fig 2 left) with three right angles and one angle $\pi/3$.

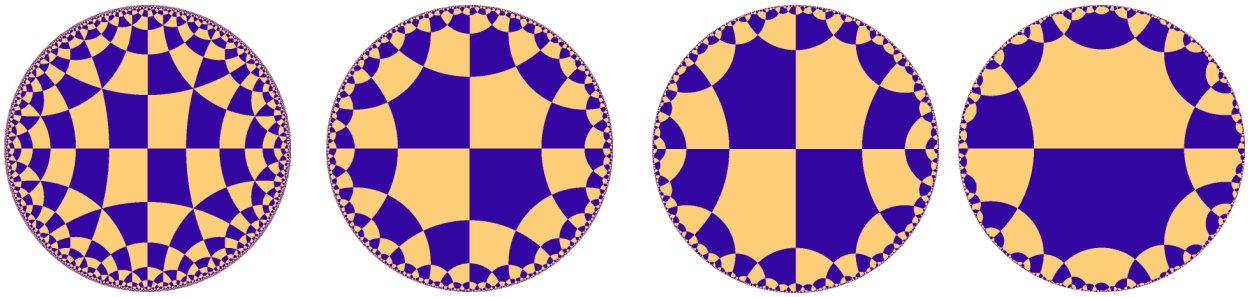

Figure 2: *Polygonal hyperbolic kaleidoscopes *3222, *22222, *222222, *22222222*

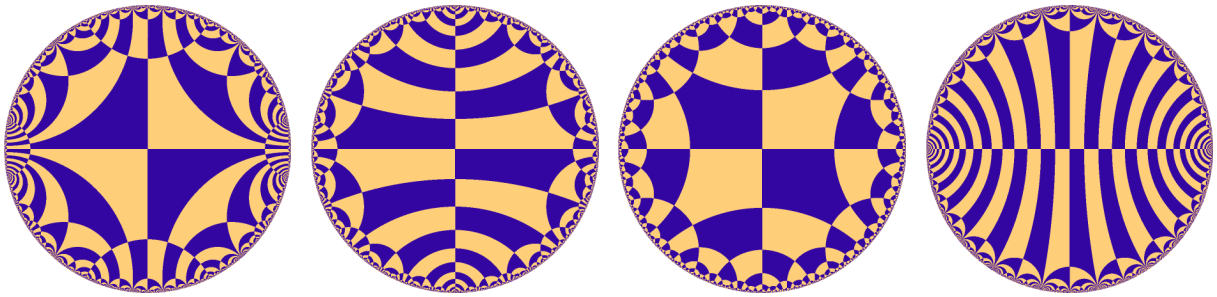

Figure 3: *Stretching pentagonal kaleidoscope *22222. Two sides of the pentagon are arbitrary.*

Let's place the polygon in *xy*-plane of the hyperbolic space presented in Klein-Beltrami ball model. Let's build three dimensional chimney with *z*-axis as it's spine and *3222 polygon in *xy* plane as its' cross section (fig.4 left). Reflections in the faces of the chimney form three dimensional hyperbolic kaleidoscope and the chimney is *fundamental polyhedron* of this kaleidoscope. This polyhedron has infinite size and infinite volume in this case. The tiling formed by this kaleidoscope in the *x-y* plane is the same as the tiling formed by two dimensional kaleidoscope. The tiling is repeated with some distortion at the upper and lower half spheres where chimney intersects the ball's boundary (fig.4 center). Let's map ball's boundary back onto the *x-y* plane using reflection in the sphere with center at (0,0,1) and radius $\sqrt{(2)}$ (stereographic projection) (fig.4 right). The tiling at the lower half-sphere is mapped into exactly the same original two dimensional tiling inside of the Poincare disk. Tiling from the upper half sphere is mapped into outside of the unit disk and is in fact inversion of the inside tiling in the unit circle.

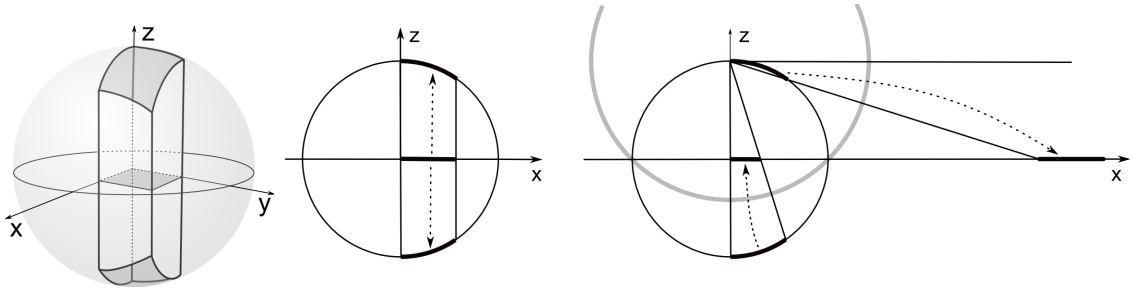

Figure 4. *Mapping the disk onto the sphere and onto the plane via stereographic projection. Drawing is done in the Klein -Beltrami model of the hyperbolic space in which faces of the chimney are euclidean planes.*

Next step is to modify the chimney to make kaleidoscope truly three dimensional. Let's rotate one face of the chimney in such a way, that dihedral angles between this face and its neighboring faces do not change. We can do this by rotating the side about common perpendicular to it's two neighboring sides. Evolution of the tiling in *x-y* plane is shown at Fig.5 and 6. The circular tiling by quadrilaterals from Fig.2 left is initially transformed into the shape with complex fractal boundary (fig 5.left). Fig. 5 center shows formation of the cusp (intersection of two sides at infinity). Further rotation will cause finite intersection of non neighboring sides of the chimney (fig 5 right). Further rotation causes formation of a new vertex of the fundamental polyhedron and closing upper entrance of the chimney. Initially this vertex is formed at infinity (fig.6 left) and moves to finite distance by further rotation. In order to have proper kaleidoscope each new edge should have proper dihedral angle (π/n). This condition puts additional constrains on the rotation angle and lengths of the sides.

If the fundamental polyhedron in the hyperbolic space has finite volume it's shape is completely defined by the set of dihedral angles and kaleidoscope is not flexible (Andreev-Thurston theorem[6]). The two dimensional pattern formed in *x-y* plane by such kaleidoscope is boring set of infinitely many infinitely small tiles, so only three dimensional visualization of such kaleidoscope makes sense, see [7] for few examples.

There are a lot of mathematical publications on the theory of symmetry groups of the hyperbolic plane and space. However, very few have illustrations going beyond the most simple (rare exceptions are [1,5]). Here we have tried to show aspects of the hyperbolic symmetries which are rarely have been visualized. Hyperbolic kaleidoscopes can be much more complex (see Fig 7 for example), and are waiting for an artistic applications.

References

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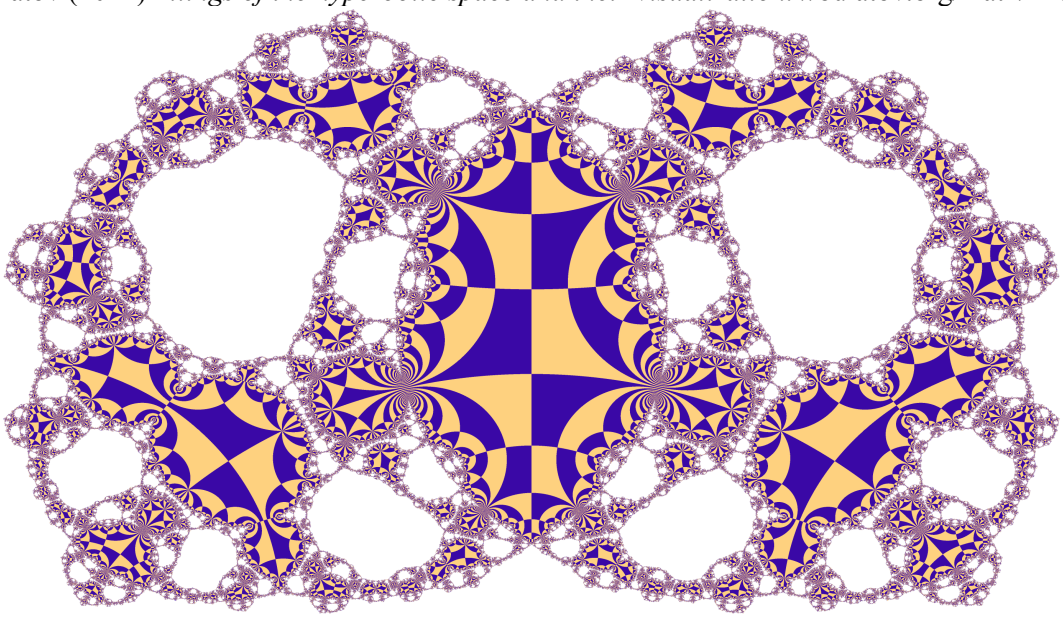

Fig 7. Kaleidoscope formed by bended octagon *22222222

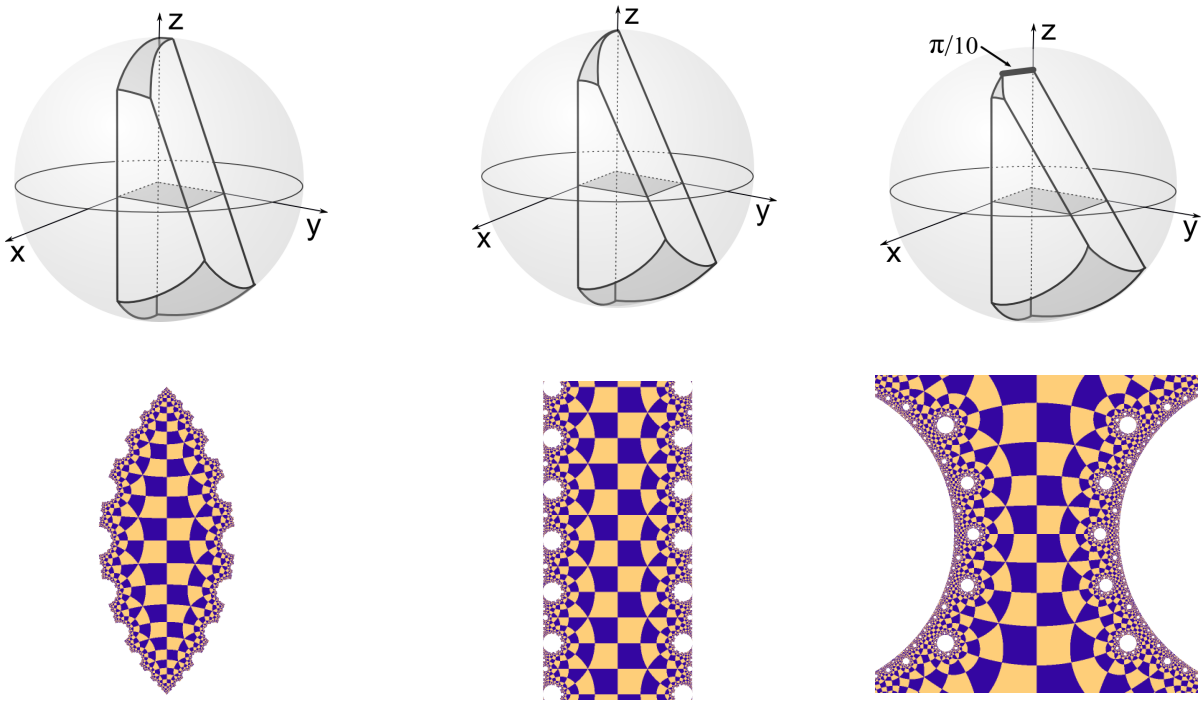

Fig 5. *Rotation of chimney's face. Tiling's boundary becomes fractal curve.*

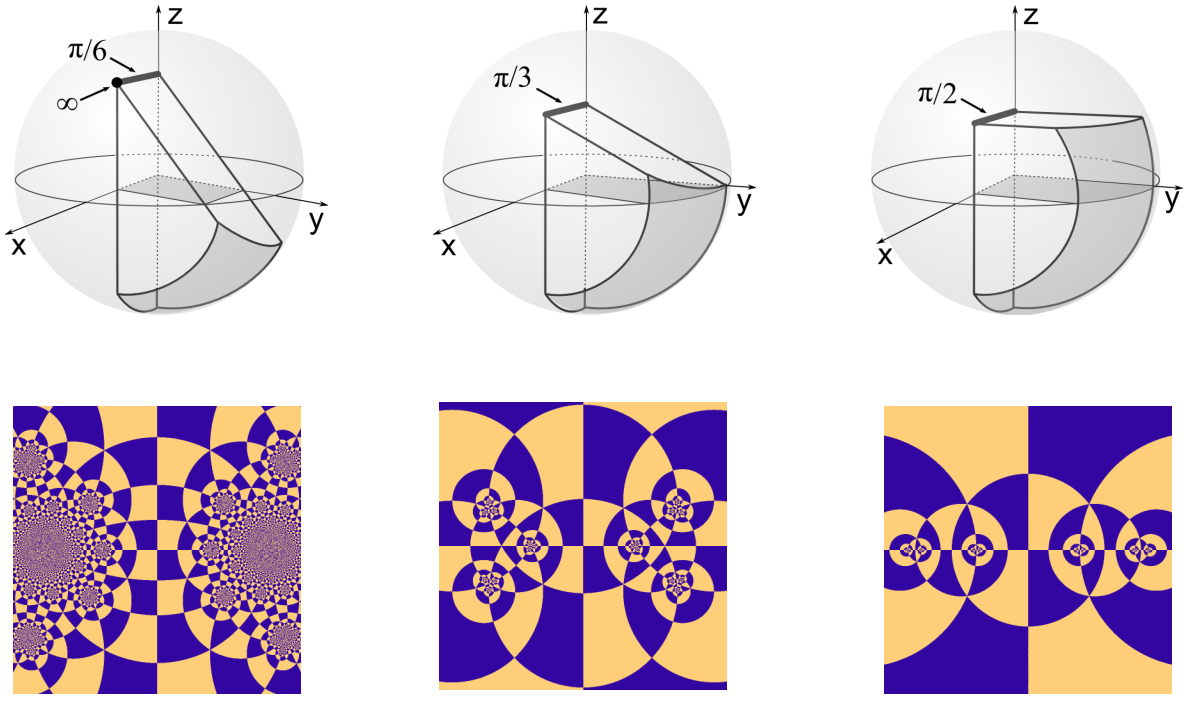

Figure 6. *Further rotation of the chimney's face. Tiling fills the whole plane.*