From Lissajous to Pas de Deux to Tattoo: The Graphic Life of a Beautiful Loop

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Abstract

The author re-implemented of a custom drawing technique—developed nearly 30 years ago for large pen plotter artwork—in order to redraw an æsthetically tuned, calligraphically widened, compound Lissajous figure, only to find it rendered a third time in an unusual medium for mathematical art: as a tattoo.

Introduction. The history of how we draw mathematical ideas accurately by machine goes back long before our current era of ubiquitous digital computation. Prior to the 1960s, when computer programmers took their first baby steps towards creating mathematical art, perhaps the most imagination-capturing method was to play with oscillatory systems, either mechanical or later electronic.

Fourier had, after all, shown in 1807 the rather startling fact that a superposition (albeit infinite) of simple trigonometric $\sin()$ and $\cos()$ functions of varying frequencies and amplitudes could be used to represent arbitrary periodic functions having nearly arbitrary shapes. The future Nobel laureate Michelson would later use an apparatus to mechanically sum and graph Fourier series.[1] Using mirrors, light sources, and tuning forks, the French mathematician Lissajous built an apparatus to help him explore the output of the simple harmonic system

$$x(t) = A_x \sin(\lambda_x t + \varphi)\,, \quad y(t) = A_y \sin(\lambda_y t) \tag{1}$$

(where time/angle t is measured in radians, the As are the amplitudes, the λs are the sinusoidal frequencies, and φ is a phase shift). A staple of laser light shows, these are known as two-dimensional Lissajous curves. They are visually elegant, typically closed, curves bounded in the plane by a rectangle centered at the origin, with sides $2A_x \times 2A_y$, where A_x and A_y are the amplitudes from equation (1).

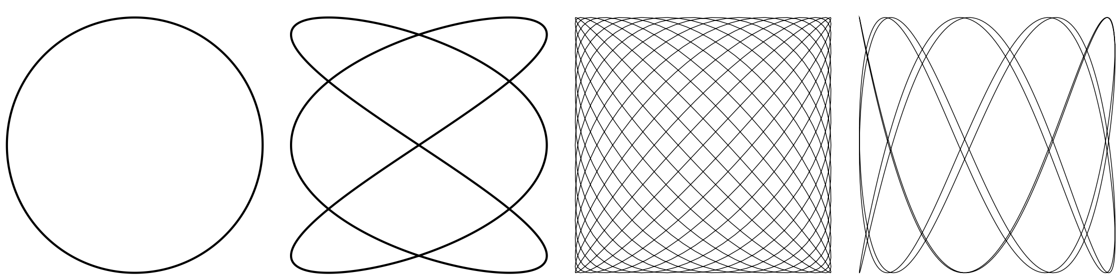

Figure 1: *Various Lissajous curves, with unit amplitudes (so they are bounded by squares, not rectangles), and using various frequency ratios. The rightmost figure also uses a phase $\varphi \neq \pi/2$.*

In the second half of the 19th century, harmonographs [1] held great fascination as a means of exploring certain compounded Lissajous figures, made more visually interesting by the effects of friction-caused

[1] Michelson also thought the machine was misbehaving due to what would later be misattributed as Gibb's Phenomenon. Although Gibbs explained the curious discontinuity to the physics community soon after Michelson's confusion, the phenomenon was first described by a 24-year-old Trinity math student, Henry Wilbraham, 50 years earlier than Gibbs. See [7].

decay in the mechanical, pendulum-based apparatus. After the 20th century invention of the cathode ray tube, electronic drawings became possible; starting in 1950, Laposky pioneered creating oscillation-based imagery, which he called "Oscillons", using photographs of electronic displays [4]; see also [3] for similar contemporary Lissajous-based work by Finkle, Franke, and Hales. Since then, commercial products depending on other kinds of mechanical oscillation, such as the well-known Spirograph™ product [5] and the lesser-known meccanograph design [1] [10], have appeared. Mathematical artists, such as Tait [9] and Moscovich [6], have further explored the sinusoidal medium with their own electro-mechanical inventions.

The etymology of *oscilloscope* is literally *a means of seeing oscillation*. As a young boy, I was fascinated by the oscilloscope my father had built in his basement electronics lab. He taught me about Lissajous figures by hooking two sin-wave generators up to its x and y inputs, and telling me to play with the knobs. This provides an immediate and wonderful visual lesson in commensurate numbers, teaching, e.g., that there might be more to the continuous number line than just fractions, or reifying that interesting things happen when integers are or are not relatively prime. In 1971, using plans published in Cundy & Rollett's *Mathematical Models* [1], as well as material found in the New York Public Library, a high-school friend and I built a harmonograph, nearly seven feet tall with heavy (lead) compound pendulums to add complex motions.

Later in the 1970s, computerization—especially digital computer graphic technology—accelerated. When Xerox PARC (Palo Alto Research Center) was developing the first experimental color laser printers around 1979, a visiting Bill Gosper pushed the technology to its limits by drawing multi-colored, Lissajous-based, polygonal textures that would cover an entire page with intricate, multi-colored lacework [2]. Since then, there has been an amazing evolution from room-sized machines to present-day (disposable!!) LCD-based cellphones. The latter are in some sense an apotheosis of a remarkable—indeed, almost unbelievable—200 years of technological progress in logic, electronics, and automated drawing of what we want to see.

Finding a Beautiful Loop. In the early 1980s, as a result of a pen plotter drawing method that I had developed (explained below), I became enamored of experimenting with graphical variations of Lissajous figures. Recalling that $\sin(\theta) = \cos(\theta - \pi/2)$, I began with the slightly altered harmonic system

$$x(t) = A_x \cos(\lambda_x t + \varphi_x)\,, \quad y(t) = A_y \sin(\lambda_y t + \varphi_y)\,. \tag{2}$$

In its simplest form, when $A_x = A_y = \lambda_x = \lambda_y = 1$, and $\varphi_x = \varphi_y = 0$, this is just the parametric equation of a unit circle (whereas equation (1) in its simplest form is a minimally interesting diagonal line). Even though mathematically one needs only a single phase difference (as in equation 1) to create the static shape of any possible Lissajous curve, by using two nominally independent φs for each coordinate, I was able to control more easily for the point $[x(0), y(0)] = [x(2\pi), y(2\pi)]$ where the loop to be drawn starts and ends parametrically. This was important for practical graphic reasons, because when a parametric loop closed on itself, it was necessary to hide the inevitable mechanical registration errors in the pen plotter's output. Regardless of mechanical error, this can also be important due to cumulative, computational round-off error.

As Figure 1 illustrates, all but the simplest Lissajous curves are self-crossing. The symmetric ones are reminiscent of the structures of various Celtic knot designs. But a Lissajous figure is either too uniform and symmetric (when the λ_x and λ_y frequencies are commensurate), or too everywhere-distributed (when the λ_x and λ_y frequencies are not), to be considered a lasting piece of *per se* visual art. I am more than happy, for instance, to celebrate bilateral symmetry as a Platonic ideal, but I find it to be an æsthetic hindrance when creating mathematical images worthy of being called art. One can change the phase φ to "rotate" the image in a manner that destroys bilateral symmetry, but the forms become too asymmetrically lopsided for my taste.

One solution was to create a compound Lissajous figure. This entails adding a second harmonic component with higher frequency and smaller amplitude, much as one adds another term in a Fourier series, or adds a secondary pendulum to the primary pendulum in a harmonograph. The new parametric system

$$x(t) = A_x \cos(\lambda_x t + \varphi_x) + B_x \sin(\omega_x t + \theta_x)\,, \quad y(t) = A_y \sin(\lambda_y t + \varphi_y) + B_y \cos(\omega_y t + \theta_y) \qquad (3)$$

resulted in significantly more complex curves, whose behavior was more difficult to predict or control. Not all the parameters are independent of one another. But there are still sufficiently many "knobs" to play with, to find/avoid shapes that attract/repel one's æsthetic sensibilities. The frequencies that work best are small, relatively prime integers: they prevent repetition and lower visual "busy-ness" so as to build low-frequency structure rather than add high-frequency texture. Even so, keeping the bounded curve from bunching into any one area of the plane is difficult to achieve. The majority of experiments one makes result in forms more akin to meaningless random scribblings than to satisfying art; Figure 2 illustrates some examples.

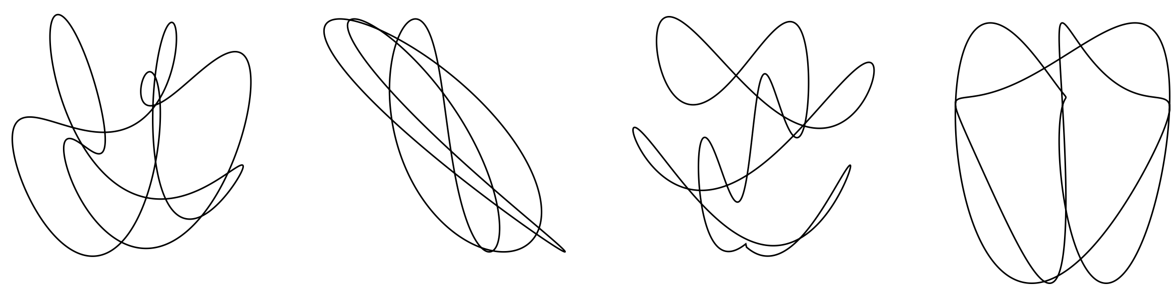

Figure 2 : *Compound Lissajous curves using various amplitudes, frequencies, and phases in equation (3).*

The sweet spot for me has always been when there is a tension and/or balance between a mathematical object's simultaneous symmetry and asymmetry. I eventually found a pair of equations that I thought did create a really interesting, visually pleasing, closed parametric curve, shown in Figure 3 for $0 \le t \le 2\pi$:

$$x(t) = -[\,5\cos(2t + \frac{\pi}{9}) + 3\sin(5t)\,]\,, \quad y(t) = -[\,5\sin(3t + \frac{\pi}{2}) + 3\cos(7t + \frac{2\pi}{9})\,]\,. \qquad (4)$$

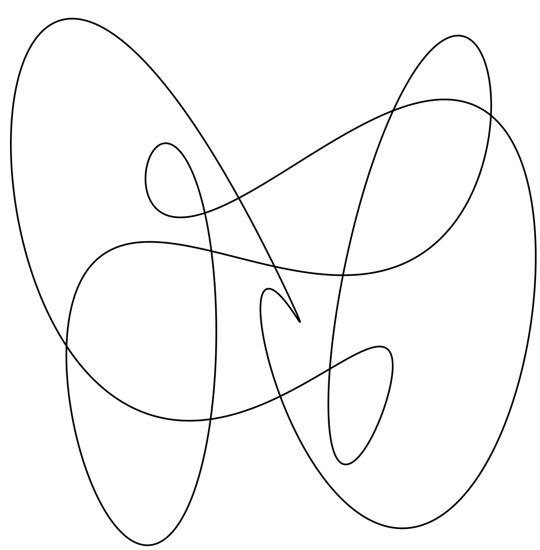

Figure 3 : *Pas de Deux curve embodying equation (4).*

Not quite rotationally or reflectively symmetric, with a single almost-but-not-quite point of derivative discontinuity near the origin, and with three pairs of somehow related, almost boisterous, interlinked loops, the figure is a captivating, graceful, æsthetically balanced form, reminiscent of a figure-skater's track in ice. Portions appear parallel, or perhaps mirrored, as if referencing each other. There's something *almost* symmetric about it, yet the turnaround point in the center simultaneously and incongruously thumbs its figurative nose at symmetry. So I think of this loop-de-loop as a parametric *pas de deux*, the result of my own æsthetic filters making a choice inherently reflecting what I care about visually, symbolically, etc.

The frequencies 2, 3, 5, and 7 in equation (4) are all small primes. The amplitudes of the secondary, higher frequency terms are smaller—but not a lot smaller—than the main lower frequency terms. Negating coordinates rotated the figure 180° so it would open upwards (not unlike the graphic/emotional/symbolic distinction between a $\vee$ and a $\wedge$). The phases were arrived at after much æsthetic "tuning" using discrete steps of $\pi/18 = 10°$. Put more algorithmically, I used conscious/unconcious hill-climbing in a sampled æsthetic continuum. Although I explored no more than a tiny portion of all the possible different amplitude, frequency, and phase combinations, I found that slight perturbations to the various parameters would harm the loop's beauty, usually in unpredictable ways. For example, the middle of Figure 4 shows the much more unbalanced and awkward curve created when φ_y in equation (4) is increased by just $\pi/18 = 10°$.

Once the mathematical form was settled upon, it was then time to play with graphic form. When giving the curve uniform thickness—as is done by modern-day line stroking algorithms, illustrated in the right side of Figure 4—I found that the important reversal point near the center loses its most salient characteristic: its sharpness. The form seems just more muddy, soft, almost . . . mushy.

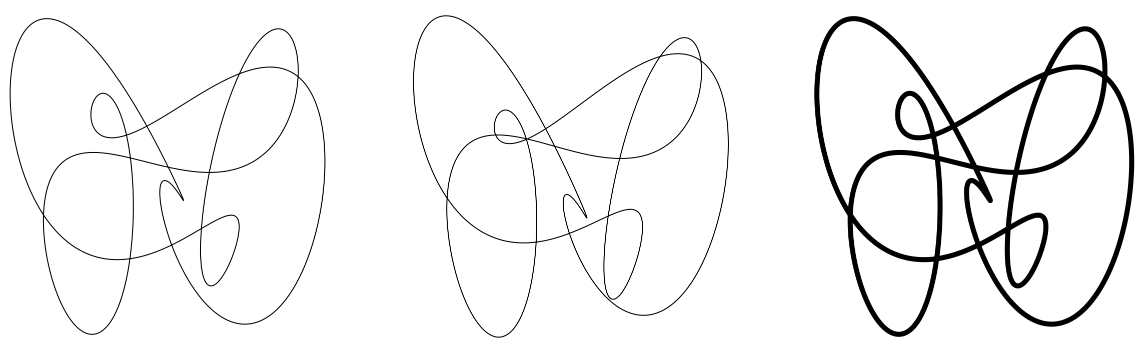

Figure 4 : *(a) Pas de Deux curve; (b) Perturbing φ_y by $+10°$; (c) Uniformly thickened Pas de Deux.*

Generalizing a pen tip's position from point to line segment. A core concept in nearly all computer graphics libraries of the last 50 years is that of positioning a pen tip within a two- or sometimes three-dimensional space. One then "sweeps" that zero-dimensional point through the space to another point, parametrically creating a line segment, circular arc, spline, or other mathematically one-dimensional graphic object. The pen tip is either up or down, with the latter condition constituting drawing. The now-ubiquitous PostScript™ language drawing model [8] is based on this design, with the additional feature that the language interpreter saves all drawing commands in a memory-based display list of piecewise-connected line or curve segments, called a *path*, prior to drawing any of them. The path is then treated as a mathematical object to which various graphical transformations—closing, smoothing, mitering, coloring, filling, etc.—are applied prior to drawing the result. Indeed, the very characters typeset on this page were designed and drawn this way. We take all this for granted now, but in the late 1970s, considerable research effort in computational geometry was spent creating algorithms that could paint the interior of an arbitrary polygon specified by just the coordinates of its vertices. Such painting was otherwise difficult to accomplish with the standard move-and-draw pen model. One could easily draw a shape's outline, but not so easily color its interior.

At the time, my medium for creating mathematical art was the pen plotter, especially one that would accept high-quality, acid-free paper and permanent ink delivered by tungsten-tipped Rapidograph™ pens. I wanted to fill shapes (especially fractal ones) and control line weight. So several years before PostScript became available, and without access to the latest research algorithms, I came up with a general technique that at least partially solved the problem. My custom pen plotter driver's fundamental primitive was not a point (x, y). Rather, it was an ordered pair of points $[\,\mathtt{a}, \mathtt{b}\,] = [(x_{\mathtt{a}}, y_{\mathtt{a}}), (x_{\mathtt{b}}, y_{\mathtt{b}})]$. The line segment connecting this pair of points was considered the virtual pen "tip." A single draw command, from [a, b] to [c, d] would thus sweep out the area of an arbitrary quadrilateral $\mathtt{a} \rightarrow \mathtt{b} \rightarrow \mathtt{d} \rightarrow \mathtt{c} \rightarrow \mathtt{a}$ (possibly crossing itself in the general case). Each successive `DrawTo` command would begin at the *endline* (analogous to endpoint) of its predecessor, creating a sequence of piecewise-connected, quadrilateral sweeps, as illustrated in Figure 5. The spacing between scan lines was controllable too, allowing interesting density effects prior to a complete fill. When $\mathtt{a} = \mathtt{b}$ and $\mathtt{c} = \mathtt{d}$, the driver would revert to standard 1-dimensional drawing between two points.

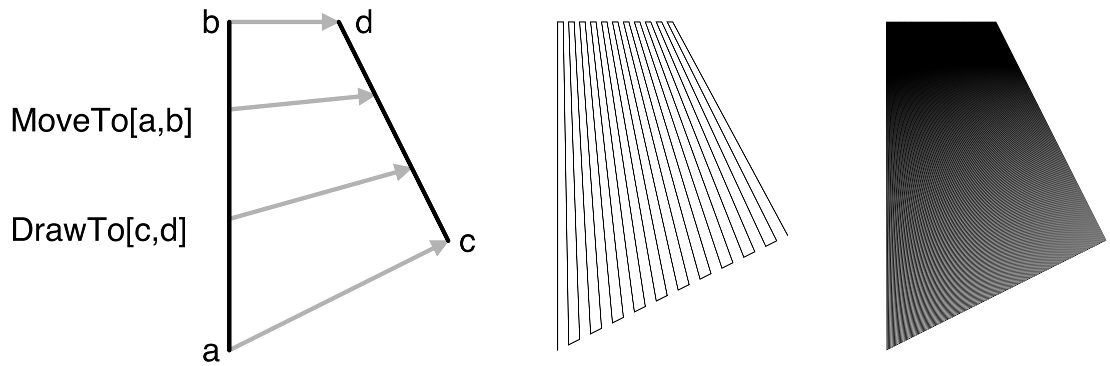

Figure 5 : *To fill a quadrilateral, one sweeps between line segments rather than points, advancing the pen no more than its line thickness, alternating scan line direction so as to never lift the pen.*

The intent was not to be calligraphic as much as it was to be general. My thinking was informed by the hypercube construction, where sweeping the vertices of an n-dimensional hypercube through space creates the next $(n+1)$-dimensional object with twice as many vertices. Hence drawing a filled unit square was nothing more than the two commands: `MoveTo[(0,0),(0,1)]; DrawTo[(1,0),(1,1)]`. And getting a pen plotter to fill the inside of a general quadrilateral was considerably easier to implement, although there were still issues related to seemlessly stitching scan lines at the shared borders of successive quadrilaterals. Better yet, it required no expensive, limited computer memory[2] for storing a complicated path in order to fill an area with ink or to draw thickened lines with mitered corners that are nothing more than sequences of edge-adjacent trapezoids. This generalized drawing model was in some ways more powerful than PostScript's, because it allowed one to vary line width parametrically *as a path was being drawn*. Other graphic constructions became much easier to accomplish, with attendant æsthetic consequences.

Calligraphic magnification from the center. Because I desired the central area of the Pas de Deux loop to be drawn in thin lines to maintain crispness, and because the outer areas were broad curves with room to be thickened, I created a sweep each of whose one-dimensional pairs $[\mathtt{a}_i, \mathtt{b}_i]$ of sequential endlines were found on an imagined line radiating from the origin and extending through and past a parametric point on the underlying curve. The dynamism of the loop is enhanced considerably using this technique, as the top of Figure 6 shows. There's more of a three-dimensional feel to it, as if black ribbons were roller-coastering back and forth through space. I called it *Rococola*, because it reminded me of the curlicue Cs in a certain soft drink's calligraphic logo (and because I like portmanteau word-play).

[2]This driver library was originally implemented on a DEC PDP-11/34 computer with 256K of memory.

Pen plotters have been almost entirely surpassed by much higher resolution, ink-jet printing of rasterized images, usually driven by PostScript. But with suitable programming, PostScript's basic path mechanism—imparting uniformly thickened lines to virtual paper—can be made to simulate the older plotter algorithms. I recreated some of those algorithms, so as to simulate the darkening of some of the overlap areas, which was another interesting effect (partial transparency) that pen plotters provided. The bottom of Figure 6 shows a close-up of the central area of the top, with the origin marked. Notice—especially in any electronic PDF version of this paper—that the entire top picture is drawn using a single, fixed-width set of connected, mitered line segments (about 8000 of them), scanning back and forth, each time advancing slightly (parametrically by $\pi/4000$) along the path of the curve. Those line segments that scan from one side of the calligraphic ribbon to the other all point at the origin, an effect that is far more pleasing, æsthetically and texturally, than what would occur in an overly general scan-conversion algorithm that might introduce horizontal or vertical artifacts unrelated to the underlying structure of the loop. There are also aliasing, Moiré artifacts in the electronic version that didn't appear in the pen plotter drawing (they are somewhat visible in the printed version). The self-intersections of this loop's widened edges preclude PostScript's two shape-filling mechanisms (`fill` or `eofill`) from creating the desired result, at least when using those edges as a path.

Unfortunately, the bane of all complex "plottery" was always (a) the fine-tipped pen getting clogged with loosened paper fibers, and (b) the pen delivering extra blobs of ink due to capillary action at the points where it was lifted from the paper to move to a new position (even more noticeable when the ink was not black). In the first case, there was nothing to do; the moment the pen stopped delivering ink, the machine would blindly proceed. The drawing—often after many hours of overnight work—would be irrevocably ruined. In the second case, though, the scan-conversion algorithm I designed to fill successive quadrilaterals guaranteed that the pen would never leave the paper, including at where pairs of quadrilaterals joined. Unfortunately, achieving this increased the chances of the pen clogging. Worse, the constant passing of the pen back and forth, advancing the edge of an inked quadrilateral, would create those loose paper fibers that awaited to wreak their havoc, should a later portion of the two-dimensional sweep draw over the same area.

So because the Pas de Deux curve is self-intersecting, especially in the outer areas where the widened "ribbons" overlap, it was almost impossible to draw in large-scale calligraphic form on a plotter, due to the pen-clogging problem. When enlarged to two or three feet in width, each drawing took several hours to finish. All but two or three of my attempts were ruined when, soon after passing over an already filled-in area, the roughened paper fibers picked up by the pen tip stopped its ink from flowing.

From pen plotter to "pin" plotter. After decades of enjoying the essentially unique calligraphic plotter drawing of Rococola on my bedroom wall, I was asked to provide a copy of my favorite loop to someone else enamored of it. Indeed, not wanting to unframe it was the motivation for my reproducing it anew (as a PostScript program), similar to how it was originally drawn 25 years ago. But not content to merely hang the artwork on a wall, my "patron" had secretly decided to include it in her "permanent collection" by tattooing herself with the image instead, see Figure 7. I did not witness the deed, but I am informed that the procedure applied ink in a similar order to how the plotter did. In this new medium, though, the algorithm for pin tip movement requires it to lift, move, and drop continually, using only capillary action to impart ink. Hence a certain loss of graphic sharpness results at the figure's edges (a likely insoluble, media-specific problem).

Conclusion. With care, it is possible to find hidden in a continuum of possibilities a compound Lissajous loop whose graceful form, when enhanced with a dynamic, calligraphic drawing technique originally used for doing arbitrary fills on pen plotters, is intriguingly—and in at least one case, painfully—beautiful.

Acknowledgements. I'm grateful to Bill Gosper for reminding me long ago that it was okay to have fun (again) with Lissajous figures, and to Ron Resch and the nascent Boston University computer graphics

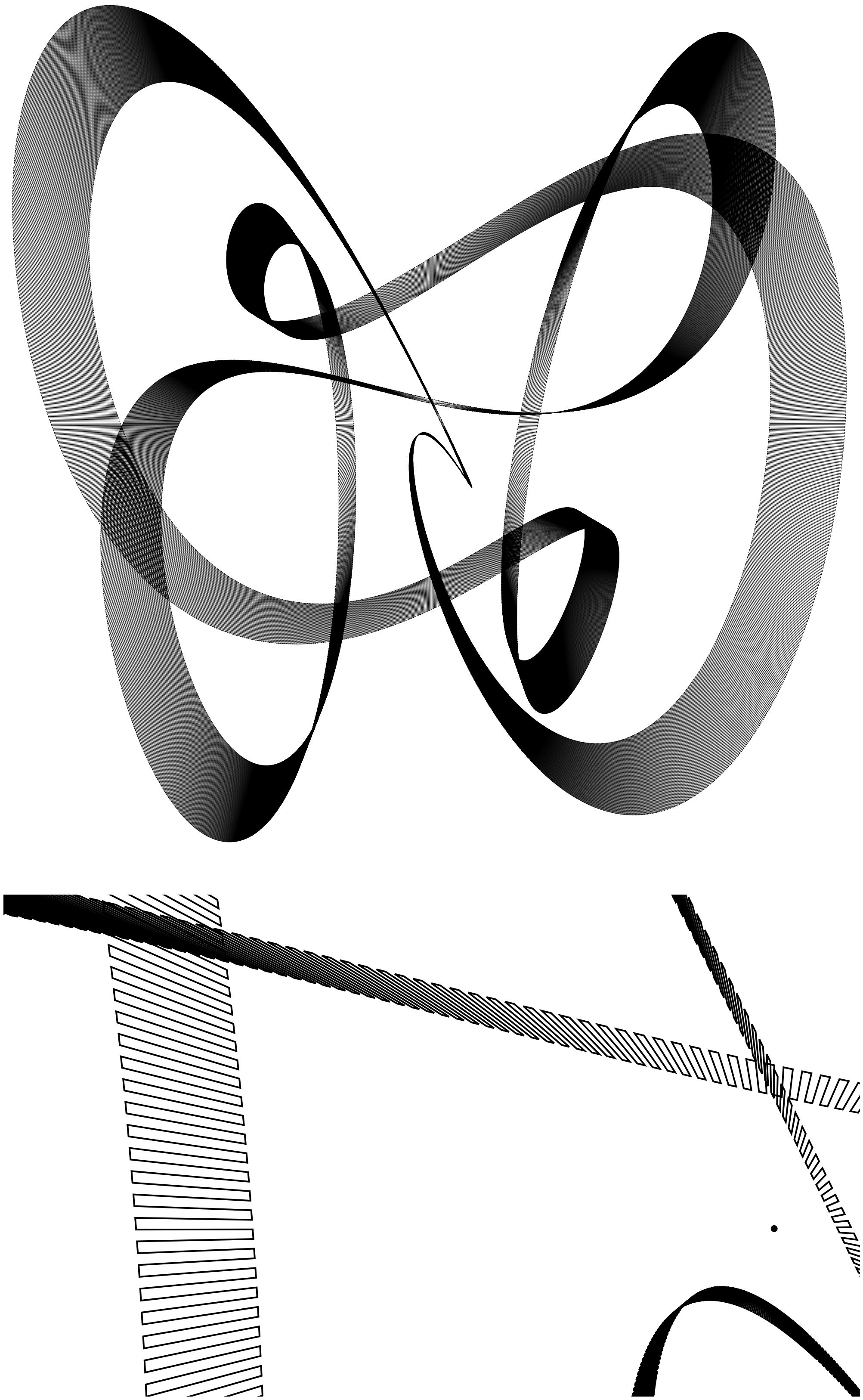

Figure 6 : *Top: PostScript simulation of a plotter drawing,* Rococola *1985. Bottom: Close up of the area near the origin (marked with a dot), which the advancing scan lines all point at.*

laboratory, whose tungsten pen tips I wore out and whose H/P plotter I nearly wore out, drawing filled shapes (fractal and otherwise) as large as possible. Finally, my recounting herein of the history, equations, and methods used would likely not have occurred without the encouragement of Alison McKenna, motivated by the continual inquiries she incurs as a result of the unique mathematical loop "skinscribed" on her back.

Figure 7 : *Tattoo,* 4×4 *inches. Medium: skin and ink.*

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[10] Video, *Meccanograph*, see `http://www.youtube.com/watch?v=b-S-tMwiPng` (as of 1/29/11).