Pitch-Space Lattices: Tonnetze and other Transpositional Networks

José Oliveira Martins
Eastman School of Music
University of Rochester
26 Gibbs St.
Rochester, NY, 14604, USA
e-mail: jmartins@esm.rochester.edu

Abstract

The paper proposes a three-fold classification for pitch-space lattices based on a music-theoretic construct known as the *Tonnetz*. It argues that applying incremental changes on some of the constructive features of the generic *Tonnetz* (Cohn 1997) results in a set of coherent and analytically versatile transpositional networks (T-nets), which brings under a focused perspective diverse pitch structures such as *Tonnetze*, *affinity spaces*, Alban Berg's "master array" of interval-cycles, and other types of networks. The paper also explores the music-modeling potential of progressive and dynamic T-nets by attending to characteristic compositional deployments in the music of Witold Lutosławski.

The Tonnetz

Figure 1a is a music–theoretic construct—known as the *Tonnetz*,—that coordinates pitch classes (pcs) distancing perfect fifths (7 semitones) and major thirds (4 semitones) along the horizontal and vertical dimensions respectively. This construct adopts a closed mod-12 group-theoretic perspective (it is a geometrical torus), and has become a central framework for the modeling of (major and minor) triadic relations in the so-called neo-Riemannian literature that has developed since the mid-1990s.[1] Richard Cohn [1] has generalized the structure of the *Tonnetz* in order to accommodate other trichordal relations and other modular cardinalities. Figure 1b presents Cohn's *generic Tonnetz*, a two-dimensional lattice that assigns an arbitrary pc-reference 0 and maps the surrounding pcs according to the pc-distance from the origin 0 under the lattice grid structured by the two axes x and y.[2]

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1 The mod12- version of the *Tonnetz* reformulates its nineteenth-century (theoretically infinite) precursor based on just intonation. The literature on neo-Riemannian theory is significantly extensive. For a historical overview and a variety of approaches to neo-Riemannian theory and analysis see the volume 42.2 of the Journal of Music Theory [2][5]. Particularly germane to this paper is [1].

2 The figure recaptures Cohn's Figure 6, p. 10 [1].

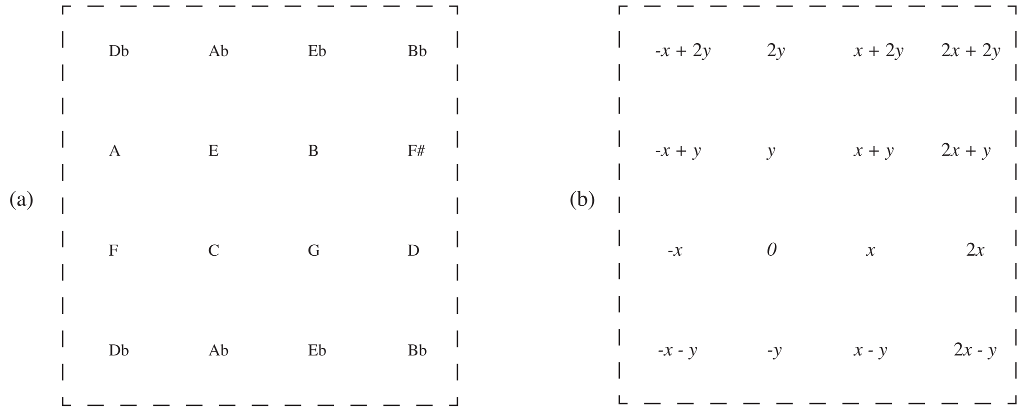

Fig. 1. (a) *Neo-Riemaniann Tonnetz*. (b) *Richard Cohn's (1997) generic Tonnetz.*

Transpositional Networks (T-nets)

The generic transpositional network. The constructive features of the generic *Tonnetz* constitute the starting point for this paper. Figure 2 introduces a generic layout for a two-dimensional lattice, or transpositional network (T-net), where a given $pc(i,j)$ (pitch class pc in position (i,j)) forms transpositional relations with adjacent pcs in the lattice. These relations are captured by the associated directed intervals $dx(i,j)$ and $dy(i,j)$ along the x- and y-axes respectively, such that $dx(i,j) = pc(i+1,j) - pc(i,j)$ and $dy(i,j) = pc(i,j+1) - pc(i,j)$.[3] Furthermore, the generalization developed here requires we consider the degree of variation of directed intervals along and across the *x*- and *y-axes*. The notation $\Delta dx \mid x$ and $\Delta dy \mid y$ refers to the change of transpositional values dx along the *x-axis*, and dy along the *y-axis* respectively; whereas $\Delta dx \mid y$ and $\Delta dy \mid x$ refer to the change of transpositional values of dx across the *y-axis*, and dy across *x-axis* respectively.[4] By defining and varying certain constraints on the four values of Δd, the paper proposes a three-fold framework of T-nets (here classified as homogeneous, progressive, and dynamic),[5] which captures important features of diverse music-theoretic pitch models such as *Tonnetze*, *affinity spaces*, Alban Berg's "master array" of interval-cycles, and other pitch constructs, and has considerable music-analytical potential for both the tonal and atonal repertoires.

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3 All the formulas throughout the paper refer to a mod-12 universe.

4 In other words: $\Delta dx \mid x = dx(i+1,j) - dx(i,j)$, $\Delta dy \mid y = dy(i,j+1) - dy(i,j)$, $\Delta dx \mid y = dx(i,j+1) - dx(i,j)$, and $\Delta dy \mid x = dy(i+1,j) - dy(i,j)$.

5 Martins [10] further defines homogeneous, progressive, and dynamic T-nets: given a pc and directed intervals at a referential position $(0,0)$, it calculates the pc and directed intervals at an arbitrary position (i,j) in the three T-nets.

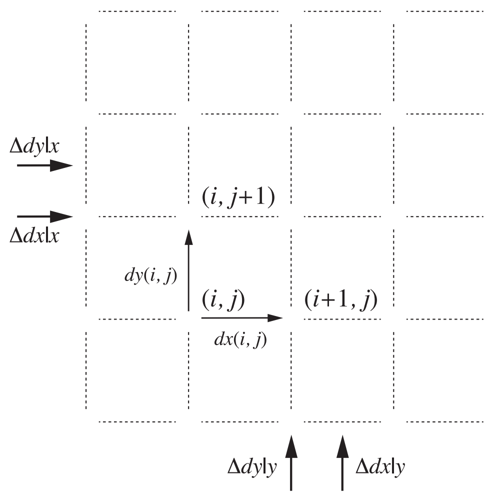

Fig. 2. *Generic layout of a trasnpositional network (T-net).*

Homogeneous T-nets. Cohn's generic *Tonnetz* can now be examined in light of the generic framework for T-nets introduced in Figure 2. Given any *x* and *y* values in a *Tonnetz*, a straight (horizontal or vertical) path constitutes an interval cycle (i-cycle), and any two parallel paths project the same i-cycle.[6] In other words, there's no change of *dx* (or *dy*) in a straight path, and there's no change of *dx* (or *dy*) in parallel paths. I refer to the *Tonnetz* as the *homogeneous T-net* in order to reflect the lack of variation in the transpositional values across the lattice. In a homogeneous T-net: $\Delta dx \mid x = \Delta dy \mid y = \Delta dx \mid y = \Delta dy \mid x = 0$. Figure 3 represents a homogeneous T-net that captures important trichordal relations for set-class (013), which is analytically relevant for a passage by J.S. Bach discussed by David Lewin.[7] In this T-net, $dx = 11$ and $dy = 2$, and the four forms of $\Delta d = 0$.

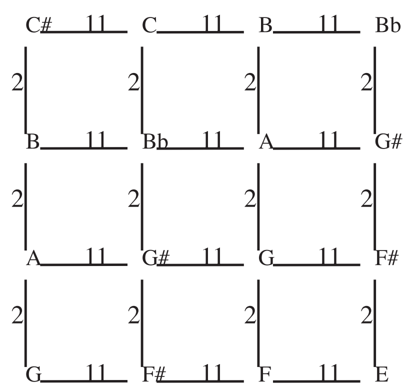

Fig. 3. *A homogeneous T-net, where* $dx = 11$ *and* $dy = 2$*, and* $\Delta d = 0$.

Progressive T-nets. The constraints that structure transpositional relations for the homogeneous T-net are now expanded to allow for a variation of transpositional values across both the *x*- and *y*-axes. The result is

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6 An interval cycle is formed by the recurrence of a given pc-interval k, $1 \leq k \leq 11$ (mod 12); this recurrence eventually returns to the stating pc. For instance <C, Eb, F#, A, (C)> (or the "diminished-seventh chord") constitutes a 3-cycle.

7 Lewin [7] for the modeling of (013) trichordal relations in a tonal passage in what amounts to a lattice where $dx = 11$ and $dy = 2$. Other homogeneous T-nets that equally model the passage are: $dx = 2, dy = 3$ or $dx = 1, dy = 3$.

a lattice; here referred to as *progressive T-net*. Figure 4a presents a familiar example of such construct, the so-called Alban Berg's "master array" of i-cycles, which coordinates all i-cycles in both vertical and horizontal dimensions.[8] Figure 4b recasts Berg's array as a T-net, emphasizing transpositional relations. Unlike in homogenous T-nets, however, in progressive T-nets parallel paths of transpositional values vary throughout the network; in the case of the Berg's array, the transpositional values increase by 1 from left to right and from bottom to top in the network; in other words: $\Delta dx \mid y = \Delta dy \mid x = 1$. In general, the constraints for a progressive T-net are $\Delta dx \mid y = \Delta dy \mid x \neq 0$ and $\Delta dx \mid x = \Delta dy \mid y = 0$.[9]

(a)

C	C	C	C	C	C	C	C	C	C	C	C	C
C	B	Bb	A	Ab	G	F#	F	E	Eb	D	C#	C
C	Bb	Ab	F#	E	D	C	Bb	Ab	F#	E	D	C
C	A	F#	Eb	C	A	F#	Eb	C	A	F#	Eb	C
C	Ab	E	C	Ab	E	C	Ab	E	C	Ab	E	C
C	G	D	A	E	B	F#	C#	Ab	Eb	Bb	F	C
C	F#	C	F#	C	F#	C	F#	C	F#	C	F#	C
C	F	Bb	Eb	Ab	C#	F#	B	E	A	D	G	C
C	E	Ab	C	E	Ab	C	E	Ab	C	E	Ab	C
C	Eb	F#	A	C	Eb	F#	A	C	Eb	F#	A	C
C	D	E	F#	Ab	Bb	C	D	E	F#	Ab	Bb	C
C	C#	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	C	C	C	C	C	C	C	C	C	C	C	C

(b)

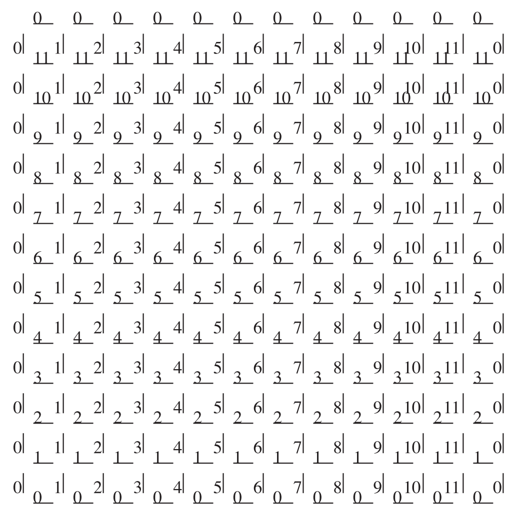

Fig. 4. (a) *Alban Berg's "master array" of interval cycles*. (b) *Berg's array represented as a progressive T-net.*

Figure 5 shows a progressive T-net, whose transpositional levels in parallel paths are incremented by 9, i.e., $\Delta dx \mid y = \Delta dy \mid x = 9$ (while $\Delta dx \mid x = \Delta dy \mid y = 0$). This progressive T-net models the closing section of Witold Lutosławski's song "Rycerze" (*Iłłakowicz Songs* 1956–57). The passage (mm. 181-199, reduced in Figure 6a) presents a series of "vertical" 12-tone chords, each being formed by the stacking of three distinct "fully-diminished-seventh chords." The transpositional framework of Figure 5 is "filled in" by the stacking of diminished-seventh chords in Figure 6b. Each stacking of three diminished-seventh chords is also a segment of what I refer to elsewhere as an *affinity space*.[10] Affinity spaces are (often non-octave) periodic constructs in which an interval pattern or modular unit (a fully-diminished-seventh chord in the case of Figure 6b) is transposed twelve times in different parts of the space. Lutosławski's closing passage is highlighted in the figure: while the 12-tone chord progression circles around all available *dy*-levels <5, 2, 11, 8, (5)> returning to its starting point at $dy = 5$, the stacking of the diminished-seventh chords does not cover all the theoretically available *dx*-levels <1, 10, 7, 4>, as the stacking does not use $dx = 4$; having done so, however, would create a repetition of the "lower" diminished-seventh chord and

8 For an historical overview of Berg's construct see Perle [11]; see also Headlam [4] for a contextualization of Berg's array in the development of Perle's cyclic-set ideas.

9 Another structure for a progressive T-net not explored in this paper is: $\Delta dx \mid x = \Delta dy \mid y \neq 0$ and $\Delta dx \mid y = \Delta dy \mid x = 0$.

10 Martins [8] introduces the notion of affinity spaces by generalizing the concept of *affinitas*, which refers to the assignment of the same local scalar pattern in different parts of the medieval scale so as to preserve modal identity. Any affinity space can also be conceived as a set of interlocked interval cycles. The generalization afforded by the affinity spaces is used to model the music of diverse composers as Bartók, Stravinsky, Milhaud, and Lutosławski. Pesce [12] traces and studies the concept of "related tones" in general, and of "affinities" in particular, from the end of the ninth century through the mid-sixteenth century. See also [3] and [6] for approaches to the modeling of interlocked cycles that are resonant with my own.

thus a tetrachordal redundancy in the 12-tone chord.[11] The transpositional levels for the stacking and succession of fully diminished-sevenths in Lutosławski's passage also ensure that no common tones are retained in adjacent tetrachords both vertically (along the *y*-axis) and horizontally (along the *x*-axis).[12]

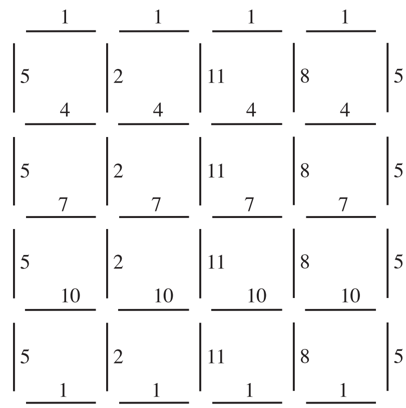

Fig. 5. *A progressive T-net:* $\Delta dx \mid y = \Delta dy \mid x = 9,\ \Delta dx \mid x = \Delta dy \mid y = 0$.

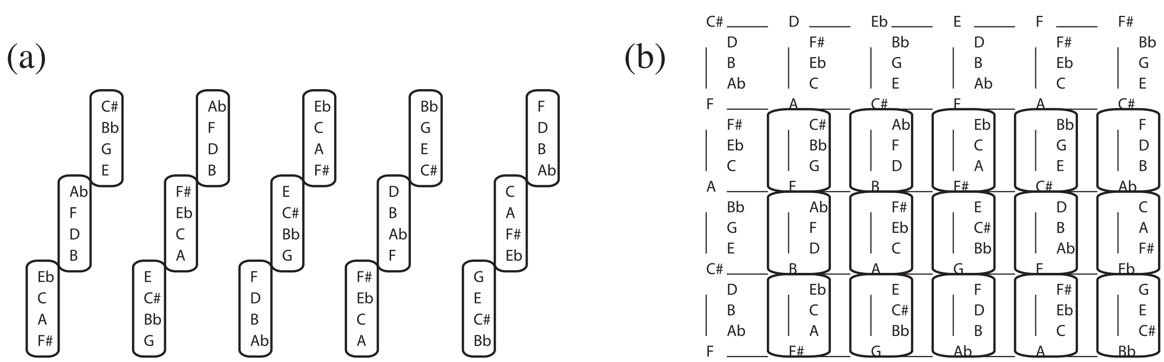

Fig. 6. (a) *Lutoslawski's 12-note chord succession (stacking of "fully- diminished-seventh chords") in the closing section of Rycerze (mm. 181-199).* (b) *Rycerze's closing section modeled by the mapping of affinity spaces within the progressive T-net.*

Dynamic T-nets. This section expands further the versatility of T-nets by modeling a constant contraction or expansion of the network in both straight and parallel paths. Such networks are here referred to as *dynamic T-nets*. Whereas in progressive T-nets the transpositional values of *dx* do not vary along the *x*-axis, and the values of *dy* do not vary along the *y*-axis, such is not the case in dynamic T-nets where the transpositional values of *dx* and *dy* do vary along their respective axes. The constraints on the rate of change for directed intervals in a dynamic T-net are thus $\Delta dx \mid x = \Delta dy \mid y \neq 0$ and $\Delta dx \mid y = \Delta dy \mid x \neq 0$.

11 The stacking of chordal structures that repeat some of the notes of the 12-tone chord does occur in *Zima* (*Iłłakowicz* Songs), where the lowest Aug-triad is rotated and repeated on top.

12 I use the term *transpositio* to refer to the interval of transposition that structures the recurrence of the modular unit, see Martins [9]; in the case of Lutoslawski's passage, *trasnpositio* corresponds to the *dy*-values. This term retains the medieval usage of referring to the transfer of a scale segment to various positions in the tonal system that preserve its modal quality (or local interval pattern), see Pesce [12].

Figure 7 presents a dynamic T-net, which models the longer central section of Lutosławski's *Postludium I* (*Three Postludes*, 1959). The transpositional changes for this dynamic T-net are $\Delta dx \mid x = \Delta dy \mid y = \Delta dx \mid y = \Delta dy \mid x = 11$. Lutosławski's passage (mm. 41–60) reduced in Figure 8 achieves a gradual contraction of pc-space by juxtaposing a series of affinity-space segments that tend towards a semitonal cluster, while retaining a constant interval of *transpositio* $t = 1$ (mod 12) (except hexachord 8).[13] Figure 9 maps the succession of affinity-space segments of the section into the dynamic T-net of Figure 7. Given the contracting aspect of the dynamic T-net from left to right and bottom to top, the affinity spaces are no longer mapped along the *y*-axes as in previously discussed T-nets, but are rather mapped in zig-zag lines along the southeast-northwest direction.

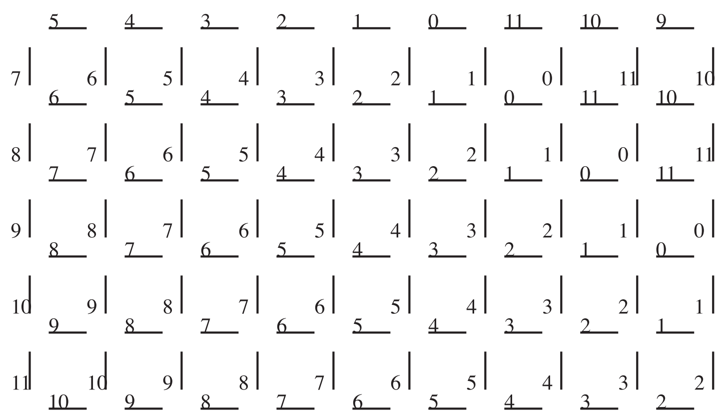

Fig. 7. *Dynamic T-net:* $\Delta dx \mid x = \Delta dy \mid y = \Delta dx \mid y = \Delta dy \mid x = 11$.

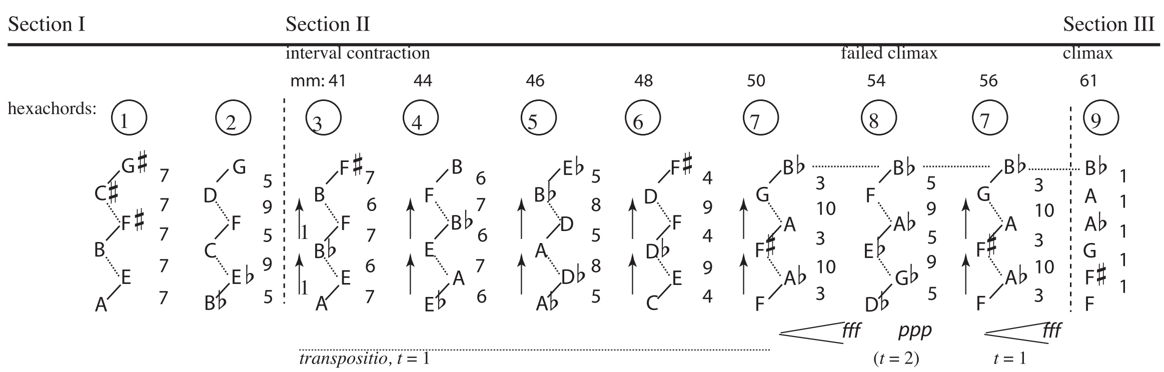

Fig. 8. *Lutosławski's Postludium I, mm. 41–60 (Three Postludes, 1959); pitch reduction of gradually contracting affinity-space segments.*

13 The value of *transpositio* =1 in the affinity space corresponds to $dy(i, j) - dx(i-1, j+1)$ or $-dx(i-1, j) + dy(i-1, j)$ in the dynamic T-net.

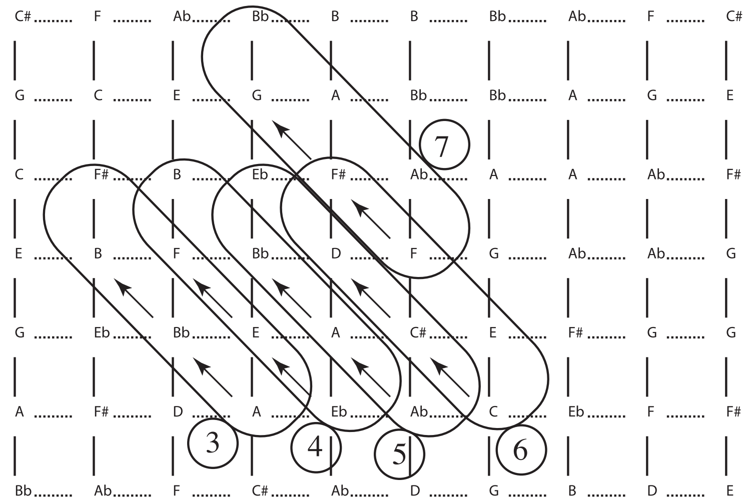

Fig. 9. *Mapping of affinity-space segments into the dynamic T-net.*

The paper proposes a framework that coordinates several models of pitch space whose constructive features rely on the concept of interval cycles and transpositional relations. This general model brings under a focused perspective diverse pitch structures such as *tonnetze*, *affinity spaces*, Alban Berg's "master array" of interval-cycles, and several types of transpositional networks, here referred to as homogeneous, progressive and dynamic T-nets. While the paper engages in incremental modifications to the constructive features of the *Tonnetz*, additional steps could be taken in the exploration of further deformations. For instance, Figure 10 presents a dynamic T-net,[14] which further deforms the changes of transpositional values while retaining a consistent structure: $\Delta dx \mid x = 1, \Delta dy \mid y = 10, \Delta x \mid y = \Delta y \mid x = 3$.

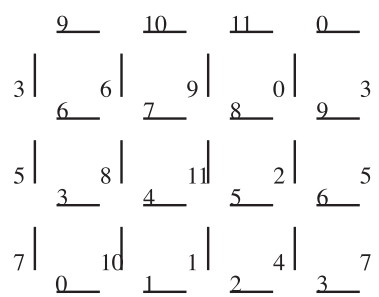

Fig. 10. *Fully dynamic T-net.*

14 This T-net could be referred to as *fully dynamic* as opposed to the previous *partially dynamic* where $\Delta dx \mid x = \Delta dy \mid y$; both dynamic T-nets, however, have the following constraint: $\Delta dx \mid y = \Delta dy \mid x$.

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