

## A Nine- and Twelve-Pointed Star Polygon Design of the *Tashkent Scrolls*

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### Abstract

In this paper we will explore one of the *Tashkent Scrolls*' repeat units, that, when replicated using symmetry operations, creates an overall pattern consisting of “nearly regular” nine-pointed, regular twelve-pointed and irregularly-shaped pentagonal star polygons. We seek to determine how the original designer of this pattern may have determined, without mensuration, the proportion and placement of the star polygons comprising the design. We will do this by proposing a plausible Euclidean “point-joining” compass-and-straightedge reconstruction.

### Introduction

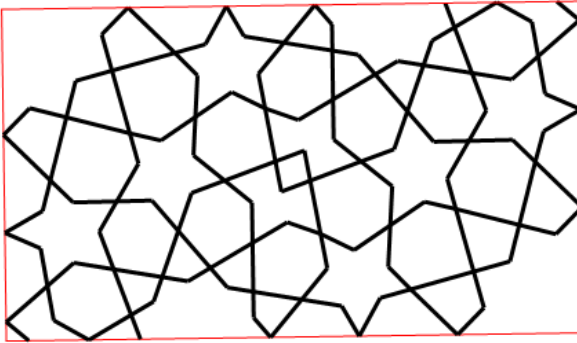
The *Tashkent Scrolls* (so named because they are housed in Tashkent at the Institute of Oriental Studies at the Academy of Sciences) consist of fragments of architectural sketches attributed to an Uzbek master builder or a guild of architects practicing in 16<sup>th</sup> century Bukhara [1, p. 7]. The sketch from the *Tashkent Scrolls* that we will explore shows only a small portion, or the *repeat unit*, of a nine- and twelve-pointed star polygon design. It is contained within a rectangle and must be reflected across the boundaries of this rectangle to achieve the entire pattern. This drawing, which for the remainder of this paper we will refer to as “T9N12,” is similar to many of the 114 Islamic architectural and ornamental design sketches found in the much larger, older and better preserved *Topkapı Scroll*, a 96-foot-long architectural scroll housed at the Topkapı Palace Museum Library in Istanbul. It is believed by Harvard professor Gülru Necipoğlu, the author of *The Topkapı Scroll – Geometry and Ornament in Islamic Architecture: Topkapı Palace Museum Library MS H. 1956*, that the repeat unit drawings of these architectural scrolls most likely “served as an *aide-memoire* for architects and master builders who were already familiar through experience with the coded graphic language used in them” [1, pages 10-12]. It is also interesting to note that T9N12 does not appear in the *Topkapı Scroll*.

How, then, using only the traditional drafting tools of the medieval period – the compass, straightedge, and set square [1, page 138] – was the T9N12 repeat unit conceptualized and created? More specifically, how did the designer of T9N12 determine, without mensuration, the proportion and placement of the polygons comprising the sketch? This paper will propose a plausible Euclidean “point-joining” compass-and-straightedge reconstruction for the design.

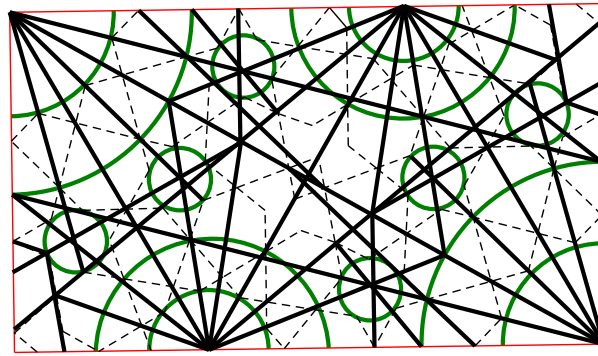
### T9N12, A Nine- and Twelve-Pointed Star Polygon *Tashkent* Design

The author has created a facsimile of T9N12 by tracing the image and producing its copy using the *Geometer's Sketchpad* software program (the electronic equivalent of the compass and straightedge [2]),

as shown in **Figure 1a**. The repeat unit contains two halves of nine-pointed star polygons along its horizontal edges and two quarters of twelve-pointed star polygons in the upper left and the lower right corners of the rectangle. Between the nine- and twelve-star polygons are six irregularly-shaped pentagonal stars, two non-regular hexagons and two overlapping arrow-like polygons that produce a parallelogram in the overlapping region they share. Underlying the black-inked drawing of the original repeat unit are faintly embossed “dead” construction lines, scratched on the paper with a pointed tool such as a compass or stylus “so as not to detract from the inked patterns they generate” [1, page 31]. **Figure 1b** shows a second facsimile created by the author of a T9N12 overlay where the inked pattern (as dashed line segments) are superimposed on the uninked “dead” construction line segments, circles and arcs (in bold). Note that the “nearly regular” nine-stars, the regular twelve-stars and the irregularly shaped pentagonal ones drawn in black ink in **Figure 1** are constructed using inscribed and circumscribed circles. The remaining polygonal shapes may be achieved by joining certain intersection points with line segments. Also, notice that the pattern in the repeat unit has 180 degree rotational symmetry about the center of the repeat unit’s rectangle (within the parallelogram of the two arrow-like shapes), which is useful in the construction of the pattern.



**Figure 1a.** Author’s reconstruction of T9N12, produced using the Geometer’s Sketchpad



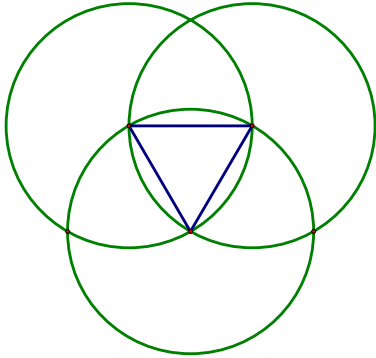
**Figure 1b.** Author’s construction of an overlay of T9N12 produced using the Geometer’s Sketchpad

The T9N12 repeat unit is interesting for several reasons. First, the major stars in the vast majority of Islamic star designs usually have an even number of points, and T9N12 has major stars with both an even number of points (the twelve-pointed stars) and an odd number of points (the nine-pointed stars). Second, the repeat unit contains nine-star polygons, presumably created from an underlying grid consisting of nine-gons, which are *not constructible* using the traditional drafting tools of straightedge and compass, so most likely an approximation was utilized to create them. Third, the position of the stars in the repeat unit is remarkable as well. Usually the major stars appear centered at either corner vertices (as the twelve-pointed stars are) or at the midpoints of the edges in the repeat units of Islamic star designs. But for T9N12, the centers of the large nine-pointed stars are not located in these standard positions.

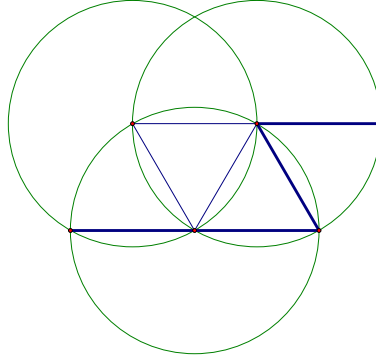
### Construction of the T9N12 Repeat Unit Rectangle

In this section, we outline a method to construct the rectangle containing the polygons that comprise the T9N12 repeat unit. Starting with a line segment of any length, construct two circles centered about the two endpoints and sharing the same radius. Using one of the points of intersection between these two circles as its center, construct a third circle through the endpoints of the line segment. Connect the centers of these three circles with line segments to form an equilateral triangle, as shown in **Figure 2a** on the following page. Extend the horizontal segments until they intersect and terminate on these circles. Also construct a segment that connects two of the points of intersection of two of the circles to form a second

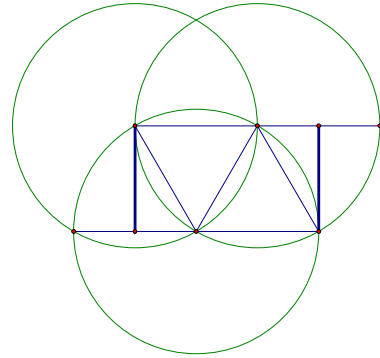
equilateral triangle, as shown in **Figure 2b**. Finally construct two perpendicular line segments through existing points that terminate at the horizontal segments, as shown in **Figure 2c**. Note that the two resulting right triangles at either end of the rectangle are 30-60-90 triangles, halves of equilateral triangles.



**Figure 2a.**

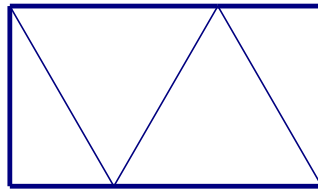


**Figure 2b.**



**Figure 2c.**

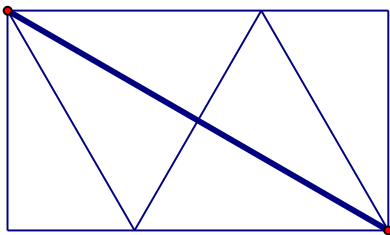
Erasing the circles and any unneeded segments yields the T9N12 repeat unit rectangle in **Figure 2d**.



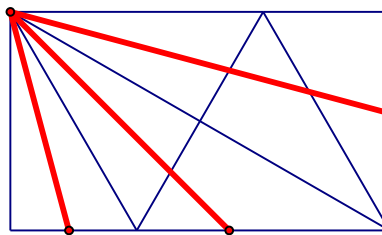
**Figure 2d.**

### Construction of the Twelve-Pointed Star Polygons

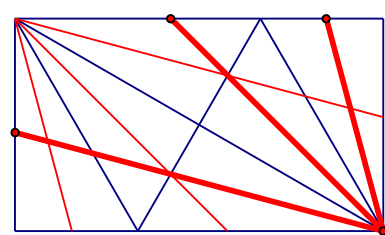
Constructing a diagonal line segment between the upper left corner and the lower right corner of the rectangle creates two additional, larger 30-60-90 right triangles and four smaller ones (congruent to the first two right triangles formed when the perpendiculars were constructed) from the halving of the equilateral triangles, as shown in **Figure 3a**. Three of the small right triangles share at their thirty degree angles the upper left corner point of the rectangle. Likewise, the lower right corner point of the rectangle is the vertex that the other three, small right triangles share at their thirty degree angles. We bisect all six of these thirty degree angles to produce the grids shown in **Figures 3b** and **3c**.



**Figure 3a.**



**Figure 3b.**



**Figure 3c.**

Next we construct congruent circles centered at the upper left corner and the lower right corner, using the existing points on the vertical edges of the rectangle, as shown in **Figure 4a** on the following page. To determine half of the vertices of a 12-gon, construct the three points of intersection formed between the

upper left circle and existing segments within the rectangle, as well as three points formed when these same segments are extended until they intersect the portion of the circle outside the rectangle. The construction of six angle bisectors produces the radii needed to construct the remaining six points of intersection that divide this circle into 12 congruent arcs. To form a regular 12-gon construct line segments between these 12 points, two adjacent ones at a time, as shown **Figure 4b**.

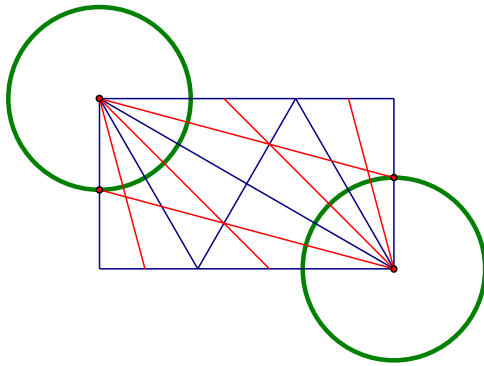


Figure 4a.

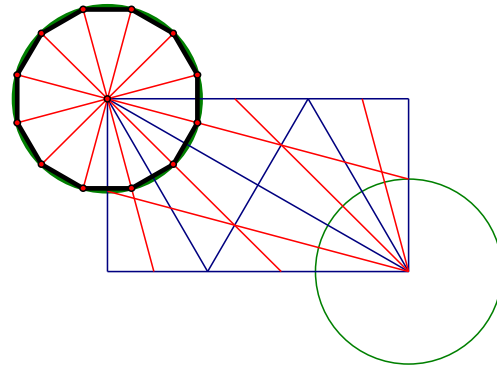


Figure 4b.

Joining every fourth point of the 12-gon creates four overlapping equilateral triangles, as shown in **Figure 4c**. (The circle is erased in this figure to more clearly illustrate the 12-gon.) By erasing some segments and highlighting others, we generate the more decorative twelve-pointed star polygon image star centered on the upper left corner of the rectangle as shown in **Figure 4d**. In a similar manner, a twelve-pointed star may also be constructed, centered on the lower right corner, as in **Figure 4e**.

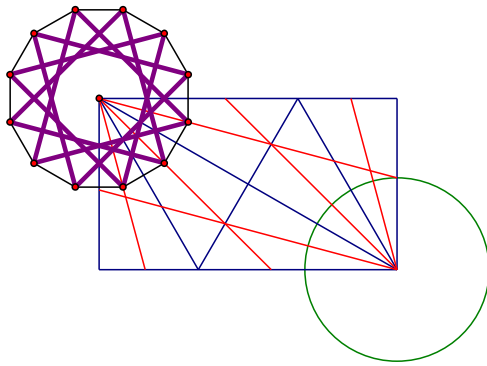


Figure 4c.

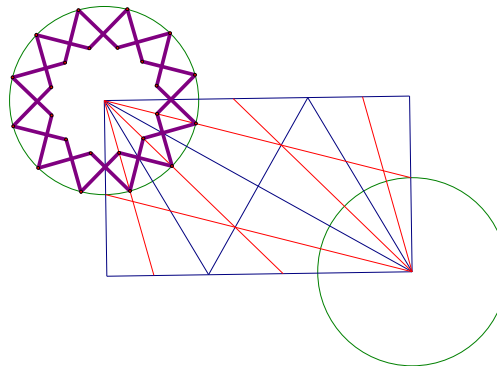


Figure 4d.

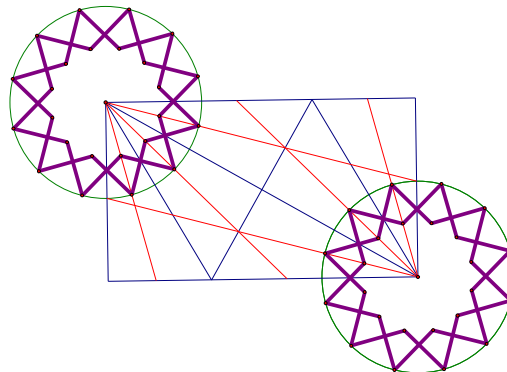
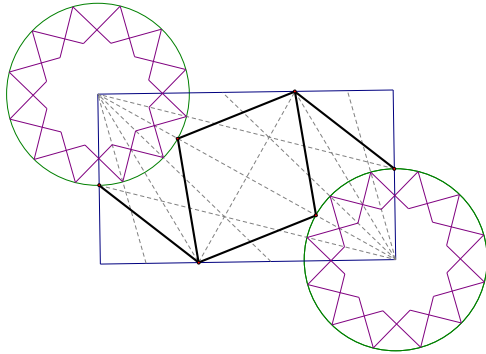


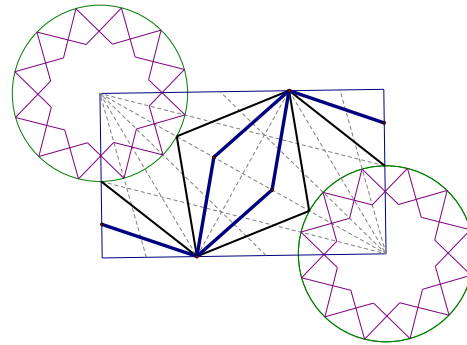
Figure 4e.

### Construction of the Nine-Pointed Star Polygons

Before we can construct the nine-pointed star, we need to draw two additional line segments – one in the lower left corner and one in the upper right corner of the rectangle – by connecting existing points on adjacent edges of the rectangle. Next, construct a large parallelogram between points of intersection of the rectangle’s diagonal with the circles containing the twelve-pointed stars and points on the horizontal edges of the rectangle, as shown in **Figure 5a**. The nine-pointed stars will be centered on the two vertices of the parallelogram that are coincident with the horizontal edges of the rectangle. Finally, we construct angle bisectors, as shown in **Figure 5b**.

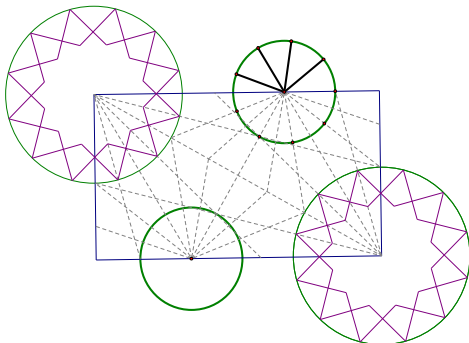


**Figure 5a.**

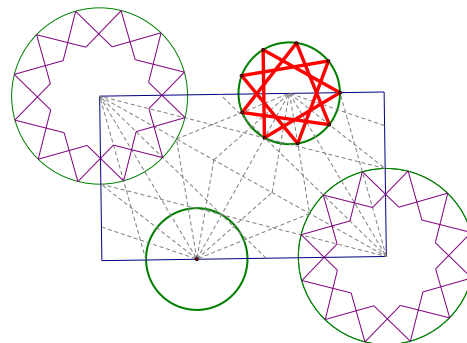


**Figure 5b.**

We are now ready to construct the nine-pointed stars. First, construct two congruent circles centered on the two vertices of the parallelogram that are coincident with the horizontal edges of the rectangle and through the points of intersection where the two segments emanating from the twelve-stars (that are closest to the rectangle’s width) intersect the horizontal edges of the rectangle. Then, construct four points of intersection (within the rectangle) between the upper right circle and the existing line segments, as shown in **Figure 6a**. Extending two existing segments yields 2 additional points on this circle. Finally construct two angle bisectors to produce the remaining two points of intersection on the circle. These nine points divide the circle into nine approximately congruent arcs. A nine-star polygon may be created by joining with line segments every third vertex of the nine found on the circle, produced from three “nearly equilateral” triangles, as shown in **Figure 6b**.



**Figure 6a.**



**Figure 6b.**

By erasing some segments and highlighting others, we generate the more decorative star polygon image in **Figure 6c**, on the following page. We repeat the same procedure to create a second nine-pointed star in the second circle, as shown in **Figure 6d**.

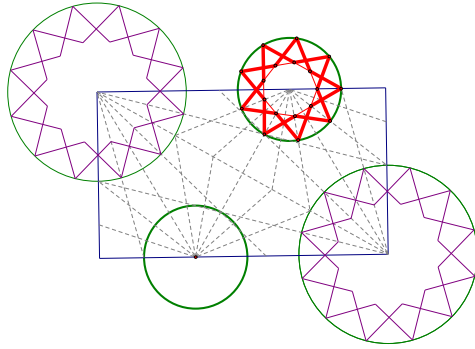


Figure 6c.

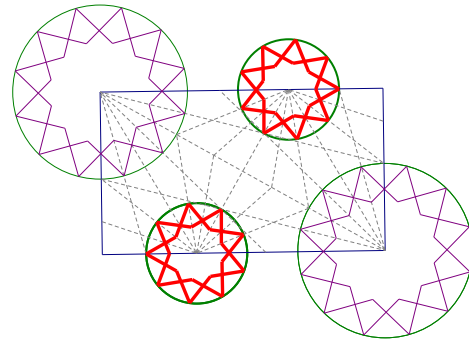


Figure 6d.

### Construction of the Polygons Found in the Interstitial Space Between the Star Polygons

With the construction of the major twelve- and nine-stars completed, we only need now to fill in the interstitial space with the irregularly-shaped pentagonal stars, hexagons and arrow-like shapes. By extending some of the line segments of the major stars until they meet existing segments (their points of intersection are denoted in **Figure 7a** by the dots), we can construct large circles centered on these stars and through these points to determine additional segments, shown in **Figure 7a** (drawn without the dots). (Note that the original circles were erased for clarity). Line segments are drawn between these points, as shown in **Figure 7b**.

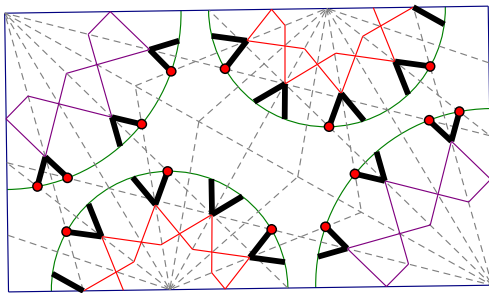


Figure 7a.

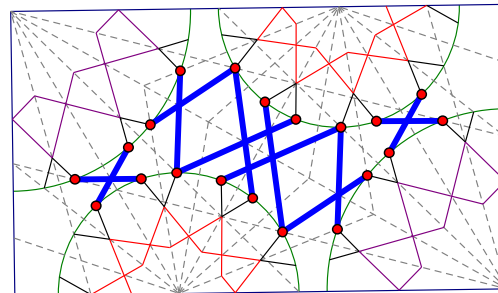


Figure 7b.

Additional points of intersection, which were constructed from some of these segments, may now be used to determine the radius of circular arcs drawn in the upper left and lower right corners of the rectangle, about the twelve-stars. The points of intersection between these arcs and the edges of the rectangle yield a few additional points and segments, as shown in **Figure 7c**. Similarly, circular arcs may be drawn about the centers of the nine-stars to determine additional points and segments, as shown in **Figure 7d**.

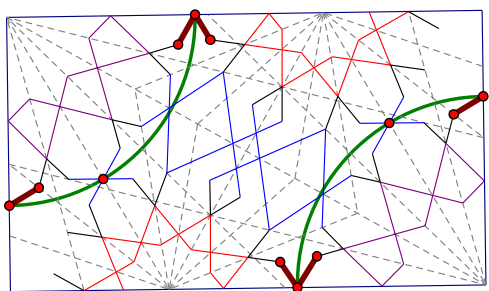


Figure 7c.

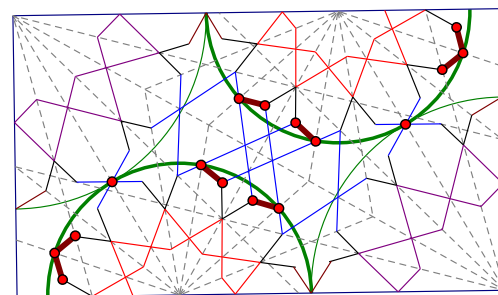
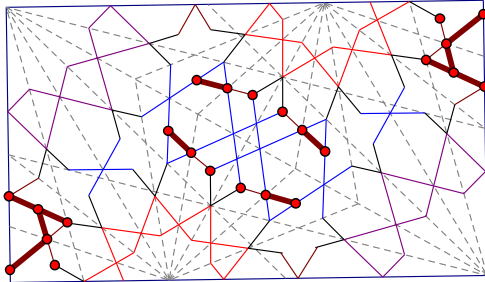


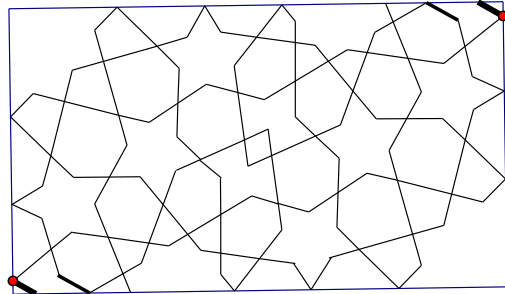
Figure 7d.



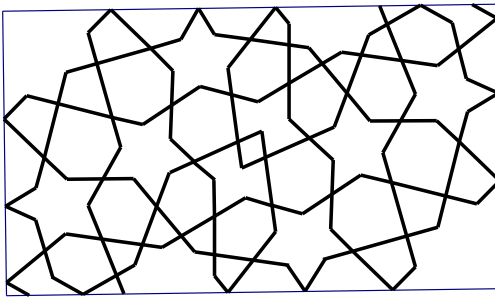
Extending some of the segments constructed in the previous step allows for the completion of the pentagonal stars and hexagons, as shown in **Figure 7e**. Lastly, short line segments are drawn through the existing points on the vertical edges and parallel to the segments of the nine-stars shown in bold in **Figure 7f**. Erasing the unneeded segments yields the completed repeat unit in skeletal form, as shown in **Figure 7g** and a colored version created using the *Microsoft Paint* program [3], as shown in **Figure 7h**.



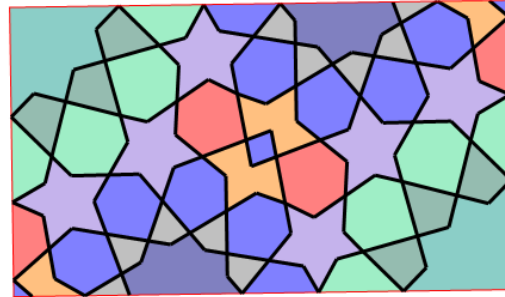
**Figure 7e.**



**Figure 7f.**

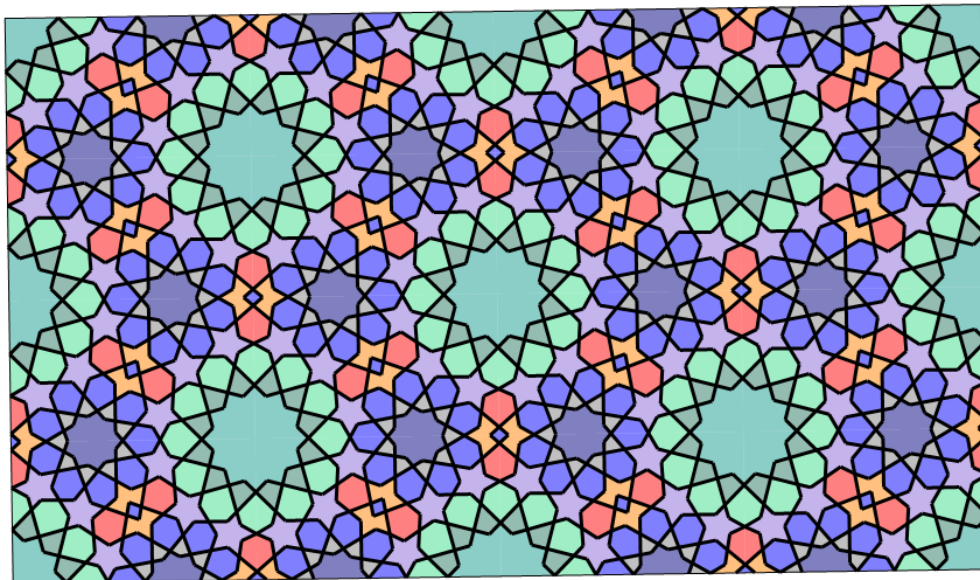


**Figure 7g.**



**Figure 7h.**

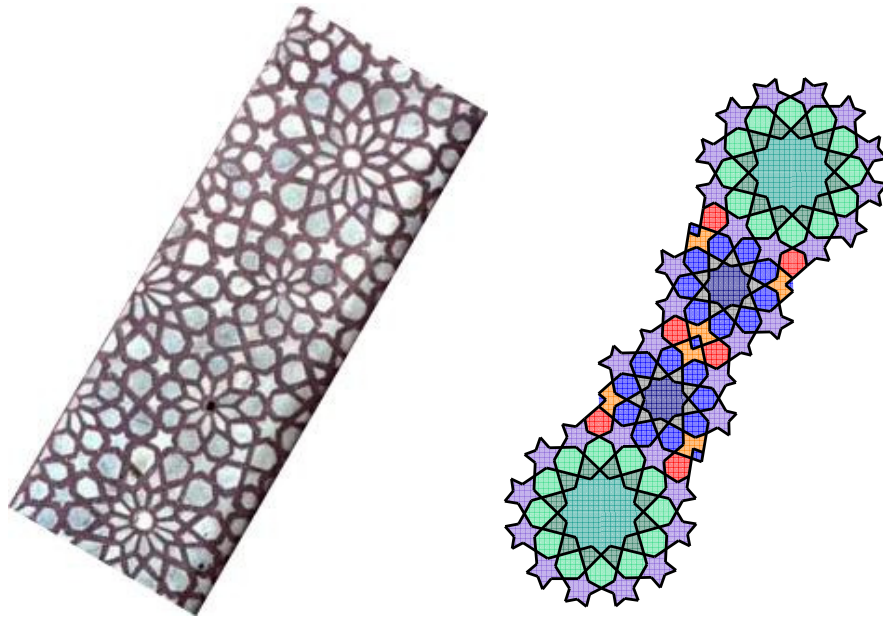
Sixteen copies of the colored repeat unit, replicated by reflection across the edges of the rectangle, are shown in **Figure 8**, below.



**Figure 8.** Sixteen copies of the repeat unit, colored and replicated by reflection across the rectangle's edges

## Discussion

What we have shown here is a plausible Euclidean “point-joining” compass-and-straightedge reconstruction for the T9N12 design. We were pleased to find at least one nearly identical extant example of the pattern generated by the replication of T9N12 repeat unit in David Wade’s collection of over 4000 images of Islamic patterns, available online [4]. The author cropped and rotated a portion of the *Ind 0729* photograph of a stone inlay and mosaic tiling at the Masjid Jami in Fatepur Sikri, India (shown below on the left of **Figure 9**) so that it would have the same orientation as the author’s reconstructed pattern. In the *Ind 0729* pattern, the interior of the stars contain additional embellishment and the arrow-shapes do not overlap to form a parallelogram, as is the case for the *Tashkent Scrolls* pattern, but otherwise the skeletal patterns are the same. The T9N12 is also similar but not identical to a pattern the author explored in “An Unusual Nine-Pointed Star Polygon Design of *La Alhambra*” [5] and which appears as Plate 120 in Bourgoin’s *Arabic Geometrical Pattern and Design* [6], a rich source of 190 Islamic patterns, first published in 1879 and based upon drawings, with analyses, of Islamic monuments in Cairo and Damascus.



**Figure 9.** *The Ind 0729 pattern from the Wade collection on the left compared with the author’s cropped Tashkent Scrolls pattern on the right*

## References

- [1] Necipoğlu, Gülru (1995). *The Topkapı Scroll – Geometry and Ornament in Islamic Architecture: Topkapı Palace Museum Library MS H. 1956*, Santa Monica, California: The Getty Center for the History of Art and Humanities.
- [2] The *Geometer’s Sketchpad* software program, available at <http://www.keypress.com>.
- [3] Microsoft *Paint* software program.
- [4] Pattern in Islamic Art: The Wade Photo Archive, available at <http://www.patternsinislamicart.com>
- [5] Bodner, B. L. (2004). “An Unusual Nine-Pointed Star Polygon Design of *La Alhambra*,” *International Society of the Arts, Mathematics and Architecture Proceedings* and also available at <http://bluehawk.monmouth.edu/~bodner/papers/2004/ISAMA04.pdf>
- [6] Bourgoin, J. (1973). *Arabic Geometrical Pattern and Design*, Dover Publications. Original 1879.