Oscillatory Solutions for Sine-Gordon Equation

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Abstract

In this paper I show how from the 2-soliton solution of the Sine-Gordon equation I create a new solution of this equation. The new solution is oscillatory, but singular.

Motivation

The Sine-Gordon equation, $u_{tt} - u_{xx} + \sin u = 0$ [1,2,5], has a *N*-soliton formula [5], which describes the interaction of an arbitrary number *N* of solitons. These types of oscillatory solutions derived from the N-soliton formula can possibly give a better understanding of singular phenomena that can happen in a system, like rogue (freak) waves for example, where the massive wave front can be understood as a singularity created by an unusual phenomenon (like an earthquake). The Sine-Gordon equation is a non-trivial model of the Field Theory as well. These types of oscillatory solutions could bring a better understanding of unusual phenomena in this field. There is a lot of study ahead, but in this paper I want to bring to attention these types of solutions, which usually bring controversy because of the singularity.

Constructing the soliton-like solutions for the Sine-Gordon equation

The Sine-Gordon equation has the analogue representation:

$$u_{xt} = \sin u \qquad (1)$$

I construct the oscillatory solutions for the Sine-Gordon equation (1) by applying a limiting process to the 2-soliton solution of the Sine-Gordon equation (1) [5]:

$$u(x,t) = -4\arg(\det(I+V)) \qquad (2)$$

where *I* is the 2×2 identity matrix and *V* is the 2×2 matrix with the following entries:

$$V_{kj} = c_j \exp(2i\lambda_j x - it/(2\lambda_j))/(\lambda_k + \lambda_j),\ k,j = 1,2,\ i = \sqrt{-1}.$$

The parameters λ_1, λ_2 and c_1, c_2 are complex parameters (they can be assigned to be real as well). In the formula (2) we consider: $\lambda_1 = 1/\mu_1 - \mu_1 \varepsilon i$, $\lambda_2 = -\overline{\lambda_1}$, $c_1 = 2\varepsilon\mu_1 \exp(1/((1/p_1)+\varepsilon) + \alpha_1\varepsilon/2)$, and $c_2 = -2\varepsilon\mu_1 \exp(-1/((1/p_1)+\varepsilon) + \alpha_1\varepsilon/2)$. Taking Taylor series expansion about $\varepsilon = 0$, and taking a limiting process as $\varepsilon \to 0$, the formula (2) becomes:

$$u(x,t) = -4\arctan(num(x,t)/denom(x,t)) \qquad (3)$$

where:

$$num(x,t) = 2\mu_1^2 \sinh(p_1)\sin((4x - \mu_1^2 t)/(2\mu_1))$$
$$denom(x,t) = -4\mu_1 x - \mu_1^3 t - \alpha_1 + 2\mu_1^2 \cosh(p_1)\cos((4x - \mu_1^2 t)/(2\mu_1))$$

The solution (3) is an oscillatory solution of (1), and it is a singular solution as well. The singularity is a result of the decaying behavior of the oscillations in space and time. The formula (3) is governed by the real parameters $\mu_1 \neq 0$, $p_1 \neq 0$, and α_1.

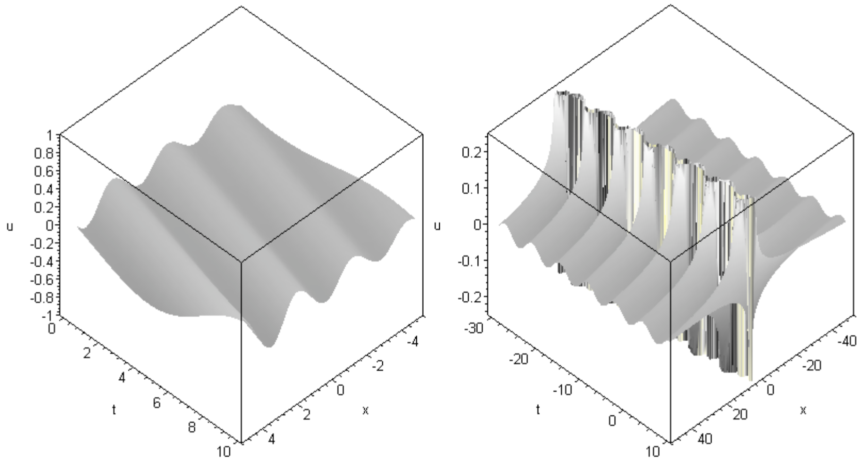
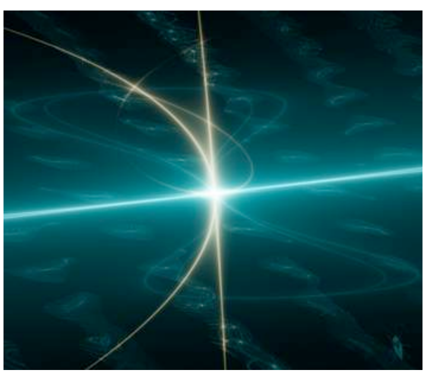
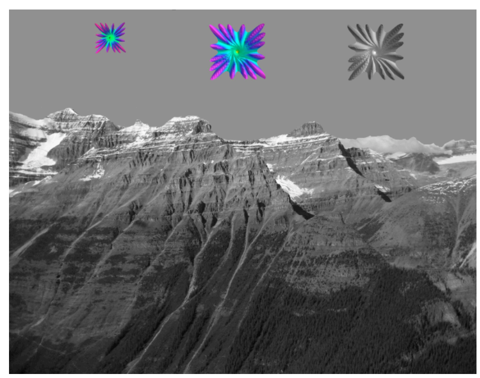


Figure: *Oscillatory solution in 3D. The first picture shows the regular part of the solution. The second picture shows the singularity. Parameters used:* $\mu_1 = -1$, $p_1 = \ln(2)$, $\alpha_1 = 50$.

Here is a beautiful picture showing oscillations near singularity and a personal picture with a twist (using the described solutions):





http://www.universaltheory.org/Singularity.html

Personal FotoShopped picture

References

[1] J. M. Ablowitz, A. P. Clarkson, *Solitons, Non-Linear Evolution Equations and Inverse Scattering*, London Mathematical Society Lecture Notes, Cambridge University Press, 1991.
[2] G. P. Drazin, S. R. Johnson, *Solitons: an introduction*, Cambridge University Press, 1989.
[3] B. B. Kadomtsev, I. V. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Soviet Physics – Doklady, Fluid Mechanics, 1970, Vol. 15, No. 6, pp.539-541.
[4] M. Kovalyov, I. Bica, *Some properties of slowly decaying oscillatory solutions of KP*, Chaos, Solitons and Fractals, 2005, Vol. 25, pp.979-989.
[5] V. Novikov, S. V. Manakov, P. L. Pitaevskii, E. V. Zakharov, *Theory of Solitons, The Inverse Scattering Method*, Contemporary Soviet Mathematics, Consultants Bureau, Plenum Publishing Corporation, 1984.