On Torsion Free Subgroups of *p*32 and Related Colored Tilings

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Abstract

In this paper, we discuss an approach in arriving at a torsion free subgroup of *p*32 - the group of orientation preserving isometries in a triangle group **p*32, using color symmetries of its related tiling. We also present the relation that exists between torsion free subgroups of *p*32 and precise colorings of a 3^p tiling.
A group is said to be torsion free if all its nonidentity elements are of infinite order.

Background

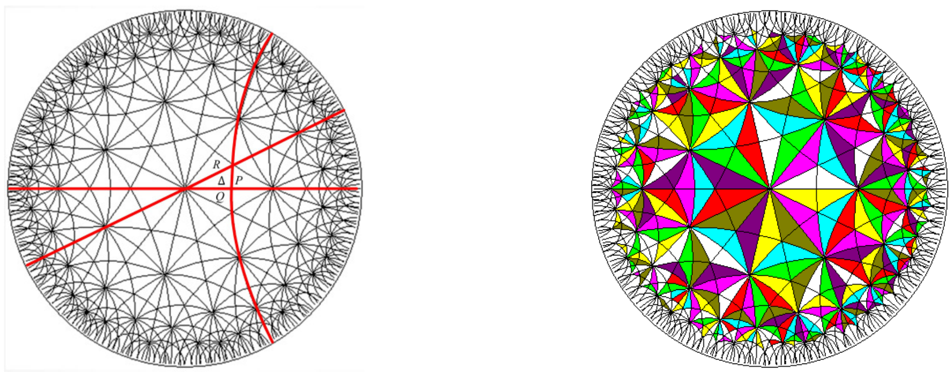
The triangle group $G = *pqr$ is generated by the reflections P, Q and R in the sides of a triangle Δ opposite its interior angles π/p, π/q and π/r where p, q, r are integers ≥ 2. The elements of G are symmetries of the triangle tiling $\mathscr{T}$ obtained by repeatedly reflecting the triangle Δ in its sides. The triangle tiling $\mathscr{T}$ is either in the S^2 (spherical), E^2 (Euclidean) or H^2 (hyperbolic) plane according as $1/p + 1/q + 1/r$ is larger, equal or smaller than 1, respectively. A subgroup of index 2 in G is the group $pqr = \langle PQ, QR, RP \rangle$ consisting of orientation preserving isometries of G with generators satisfying $(PQ)^r = (QR)^p = (RP)^q = e$. When $q=3$ and $r=2$, the group $G=*p32$ occurs as the symmetry group of the regular 3^p tiling of the plane by equilateral triangles meeting p at a vertex.

In this paper, we present an approach to arrive at a torsion free subgroup of the group *p32*. By a torsion free subgroup, we mean a subgroup all of whose nonidentity elements are of infinite order. The geometrical interest of such a subgroup is that when the points on the plane of the tiling are identified using the elements of the subgroup, the result is a manifold. We will also present the relationship between precise colorings of the 3^p tiling using p colors (p-colorings) and torsion free subgroups of *p*32. By a *precise p-coloring* of the 3^p tiling, we mean a coloring where at each vertex the p triangles around the vertex have different colors, that is the p colors appear exactly once in each vertex of the tiling.

Subgroups of the Triangle Group **pqr* and Colorings of $\mathscr{T}$

The problem on finding the torsion free subgroups of triangle groups is part of the more general problem we have been addressing on the determination of the subgroup structure of triangle groups. The approach presented in [3, 4] is via color symmetry theory. Specifically, in determining the subgroups of a subgroup H of a triangle group, we use the correspondence between the index n subgroups of H and n-colorings of the H-orbit of Δ where the elements of H effect a permutation of the n colors and H is transitive on the set of n colors.

To illustrate this idea, consider the tiling $\mathcal{T}$ given in Fig. 1(a) which is obtained by reflecting the triangle Δ with interior angles $\pi/7$, $\pi/3$ and $\pi/2$ along its sides. The group generated by reflections P, Q and R along the sides of Δ is the triangle group *732. Note that the tiling $\mathcal{T}$ is the *732 orbit of Δ. The coloring shown in Fig. 1(b) is an 8-coloring of $\mathcal{T}$ and the group *732 acts transitively on the 8 colors. It can be verified that the group of symmetries fixing color white is generated by the reflections RQR and P, and the 3-fold rotation $QRQPRQRQ$. The group $<RQR, P, QRQPRQRQ>$ is an index 8 subgroup of *732.



(a) (b)

Figure 1.(a) *A triangle tiling $\mathcal{T}$ on the Poincare' model of the hyperbolic plane by a fundamental triangle Δ with interior angles $\pi/7, \pi/, \pi/2$, together with the reflection axes of P,Q,R. (b) A *732 transitive 8-coloring of $\mathcal{T}$. The tilings are drawn and colored using Coloring the Hyperbolic Plane (CHP) [1] software.*

Let H be a subgroup of a triangle group. To arrive at a subgroup of H of index n, we construct n-colorings of the H-orbit O of Δ using a set $C = \{c_1, c_2, \dots, c_n\}$ of n colors with the property that all elements of H effect permutations of C and H acts transitively on C. Denote the colors $c_1, c_2, \dots, c_n$ respectively by $1, 2, \dots, n$. Now, for each such coloring of O, a homomorphism $\pi\colon H \to S_n$ is defined where for each $h \in H$, $\pi(h)$ is the permutation of the colors in O effected by h. If $H = < h_1, h_2, \dots, h_m >$, π is completely determined when $\pi(h_1), \pi(h_2), \dots, \pi(h_m)$ are specified. We call the set $\{\pi(h_1), \pi(h_2), \dots, \pi(h_m)\}$a permutation assignment that gives rise to a subgroup S of H of index n. The subgroup S consists of all elements of H that fix a specific color in the n-coloring [3].

For example, the permutation assignment $\{\pi(P), \pi(Q), \pi(R)\} = \{(3,4)(5,6)(7,8), (1,2)(5,7)(6,8), (1,3)(2,5)(7,8)\}$ gives rise to the index 8 subgroup $<RQR, P, QRQPRQRQ>$ of *732 discussed above. The colors white, yellow, pink, blue, green, violet, gold and red in Fig. 1(b) are assigned the numbers 1, 2, … , 8, respectively.

To determine if a given subgroup of H is torsion-free, we are going to refer to its corresponding permutation assignment as a starting point. We discuss this approach in the next section.

Deriving a Torsion Free Subgroup of *pqr*

In identifying torsion-free subgroups of pqr, let us consider a permutation assignment $\alpha = \{\pi(PQ), \pi(RP), \pi(QR)\}$ which gives rise to an index m subgroup M of pqr. Suppose y is any of the generators PQ, RP, QR of pqr. If $\pi(y)$ fixes a number $i \in \{1, 2,\dots, m\}$, then $zyz^{-1} \in M$ for some $z \in pqr$. Moreover, if $\pi(y)$ is of order b, then $y^b \in M$. It is a well known result that an element $w \in pqr$ is of finite order if and only if it is a conjugate to a power of either PQ, RP, QR [7]. We can then say that a permutation assignment α which gives rise to a subgroup M of pqr is torsion-free when, for each generator y of pqr, $y \in \{PQ, RP, QR\}$,

$\pi(y)$ does not fix any number i and the order of $\pi(y)$ is equal to the order of y. To satisfy both conditions, if y has order d, $\pi(y)$ should be a permutation consisting of a product of m/d disjoint d-cycles. Such a permutation assignment α with this property is called a semi-regular permutation assignment.

We state the following result from [8].

Theorem 1. A subgroup M of pqr is torsion-free if and only if the permutation assignment α that gives rise to M is a semi-regular permutation assignment.

For example, in obtaining a torsion free subgroup of $p32$ of index m, we must have a semi-regular permutation assignment $\alpha = \{\pi(PQ), \pi(RP), \pi(QR)\}$ where $\pi(PQ)$ is a permutation which is a product of $m/2$ disjoint 2-cycles, $\pi(RP)$ is a product of $m/3$ disjoint 3-cycles, and $\pi(QR)$ is a product of m/p disjoint p-cycles.

As a first illustration, we derive a torsion free subgroup of 632, the group of orientation preserving isometries in the triangle group *632. From a computer generated list of permutation assignments corresponding to the subgroups of 632, a torsion free subgroup of index 18 in 632 arises from the permutation assignment α where $\pi(PQ) = (1,13)(2,7)(3,15)(4,9)(5,17)(6,11)\ (8,18)(12,16)(10,14)$; $\pi(RP) = (1,18,7)(2,12,15)(3,14,9)(4,8,17)(5,16,11)(6,10,13)$ and $\pi(QR) = (1, 2, 3, 4, 5, 6,)(7, 8, 9, 10, 11, 12)(13, 14, 15, 16, 17, 18)$. Note that α is a semi-regular permutation assignment since $\pi(PQ)$, $\pi(RP)$ and $\pi(QR)$ are permutations which are products respectively of 9 disjoint 2-cycles, 6 disjoint 3-cycles and 3 disjoint 6-cycles. An 18-coloring of the 632 orbit of the triangle Δ' with interior angles $\pi/6$, $\pi/3$ and $\pi/2$ is given in Fig. 2. The numbers 1 to 18 are used to denote the colors. The torsion free subgroup of index 18 in 632 is $< RPQRQRQPRQ, RQRPQRQRQP>$, a group of translations. When the points on the plane are identified using the translations in this subgroup the resulting manifold is a torus. It is interesting to note that this subgroup of index 18 in the group 632 also arises from a precise coloring of the 3^6 tiling. We will discuss how a torsion free subgroup of the group $p32$ results from a precise coloring of a 3^p tiling in the next section.

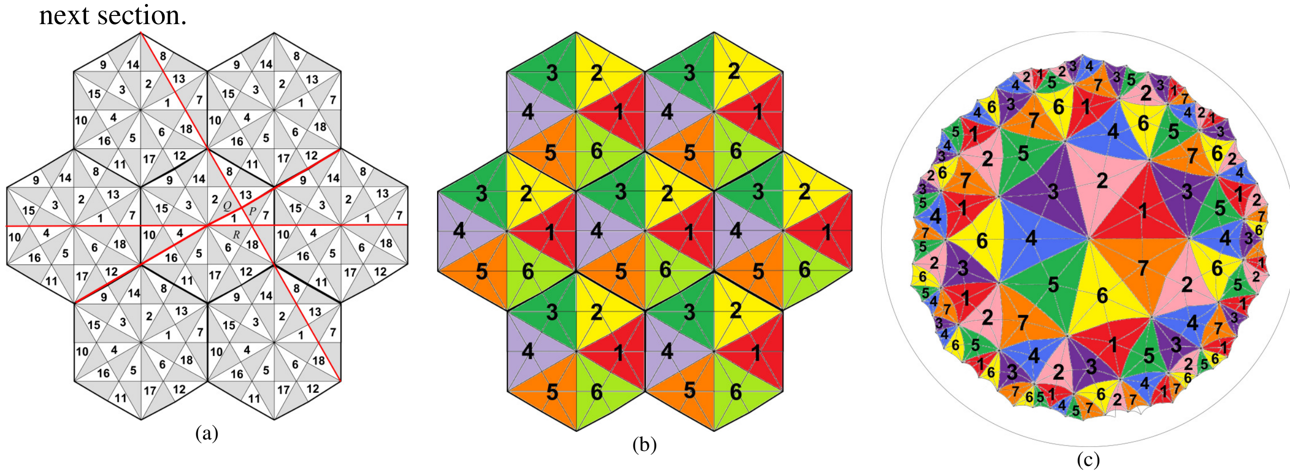


Figure 2: *(a) An 18-coloring of the 632 orbit of the triangle Δ' with interior angles $\pi/6$, $\pi/3$ and $\pi/2$, together with the reflection axes of P, Q and R. (b) A precise 6-coloring of the 3^6 tiling. (c) A precise 7-coloring of the hyperbolic 3^7 tiling.*

Torsion Free Subgroups of the Group $p32$ and Precise Colorings of the 3^p Tiling

A 3^p tiling of the plane is a tiling by equilateral triangles meeting p at a vertex. Its symmetry group is the triangle group $*p32$ which is also the symmetry group of the tiling by triangles with interior angles π/p, $\pi/3$ and $\pi/2$. In Fig. 2(b)-(c) we exhibit precise colorings of the 3^6 and 3^7 tiling. In each of these precise colorings it can be shown that the subgroup that fixes the colors in the given coloring is a normal torsion free subgroup of 632 and 732, respectively.

Consider a precise coloring of the 3^p tiling using p colors. Under the action of $H = p32 = \langle PQ, RP, QR \rangle$ on the set $C' = \{c_1, c_2, \ldots, c_p\}$ of colors, a homomorphism $f: H \to S_p$ is determined where for each $h \in H$, $f(h)$ is the color permutation induced by h. From this, we have $H/K \cong f(H)$ where K = the kernel of the homomorphism f. This subgroup K which consists of elements of H that fix the colors in the given precise coloring is a torsion free subgroup of H. Note that K does not contain nonidentity elements of finite order. Given a precise coloring of the 3^p tiling, a p-fold rotation with center at a vertex of the 3^p tiling, a 3-fold rotation with center at the center of an equilateral triangle of the 3^p tiling or a half-turn with center lying on the midpoint of an edge of the 3^p tiling has the property that it does not fix all colors of the tiling, that is there is at least one color that is sent to another color by the rotation. Thus the nonidentity elements of finite order in $p32$ are not in K and so K is torsion free.

We state this result in the following theorem.

Theorem 2. In a precise coloring of the 3^p tiling with p colors, the group K consisting of symmetries of $p32$ that fix the colors in the given coloring is a torsion free subgroup of $p32$.

For the precise coloring of 6 colors $c_1, c_2, \ldots, c_6$ shown in Fig. 2(b), the image of H under the homomorphism $f: H \to S_6$ is generated by $f(PQ) = (c_1, c_2)(c_3, c_4)(c_5, c_6)$, $f(RP) = (c_2, c_6, c_4)$ and $f(QR) = (c_1, c_2, c_3, c_4, c_5, c_6)$. This group has order 18 and is isomorphic to $D_3 \times Z_3$. The subgroup K fixing the colors of the precise coloring is the group $\langle RPQRQRQPRQ, RQRPQRQRQP \rangle$ which is the torsion free subgroup of index 18 of 632 obtained in the previous section.

Now, to derive a torsion free subgroup of 732 using the above notion of a precise coloring, we start with the precise coloring of 7 colors $c_1, c_2, \ldots, c_7$ shown in Fig. 2(c). The image of H under the homomorphism $f': H \to S_7$ is generated by $f'(RQ) = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$, $f'(RP) = (c_2, c_3, c_7)(c_4, c_5, c_6)$ and $f'(PQ) = (c_1, c_7)(c_3, c_6)$. Using GAP, it can be verified that this group has order 168, and is a subgroup of S_7. This is actually the Klein group of order 168 and is isomorphic to PSL(2,7). This implies that 732 has a torsion free subgroup of index 168. As a matter of fact, when the points on the plane are identified using the elements of this torsion free subgroup the result is a manifold which is a 3-holed torus [11]. By Theorem 1, corresponding to this torsion free subgroup of index 168 is a semi-regular permutation assignment $\alpha' = \{\pi'(PQ), \pi'(RP), \pi'(QR)\}$ such that $\pi'(PQ)$ is a product of 84 disjoint 2-cycles, $\pi'(RP)$ is a product of 56 disjoint 3-cycles and $\pi'(QR)$ is a product of 24 disjoint 7-cycles.

The problem of arriving at precise colorings of triangular tilings is discussed in [9]. In [5], precise colorings of the 3^8, 3^9, 3^{10} tilings may be found. For each coloring, the subgroup K consisting of the color preserving elements of 832, 932, (10)32 respectively is a torsion free subgroup. In each instance it is interesting to determine the result when the points on the plane are identified using the elements of K.

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