A Fractal Celtic Key Pattern?

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Abstract

A particular Celtic key pattern, dating from around 1000 AD, displays conspicuous self-similarity, which can be developed in a natural way to any depth of recursion. Such self-similar structures, although rare, are not unknown in the medieval period, but it is unlikely that there was a well-developed concept of recursion at that time.

Celtic Key Patterns

Along with the more widely known interlace, so-called Celtic key patterns are common in the manuscripts and stone monuments created in the British Isles during the second half of the first millennium of the Common Era. Their classification by Romilly Allen [1], which is still standard among archaeologists, has become much easier to access since it was reprinted [2], and it provides a lot of useful analysis, but all such attempts at pattern classification have the intrinsic problem of deciding what is essential, and what is merely variation. It is worse with interlace, because there are usually many ways to choose the basic units, and alternatives to Romilly Allen's attempt have been proposed [3]. Peter Cromwell avoids the problem in his recent survey [4], based on knot types, because it is restricted to small knots. Sometimes the more straightforward classification of key patterns can obscure properties that are more mathematically significant.

Figure 1 shows a drawing [5] of the pattern from the left side of a font in Penmon Priory Church, Anglesey, Wales, which is believed to have been originally a cross-base, dating from late 10th- early 11th century.

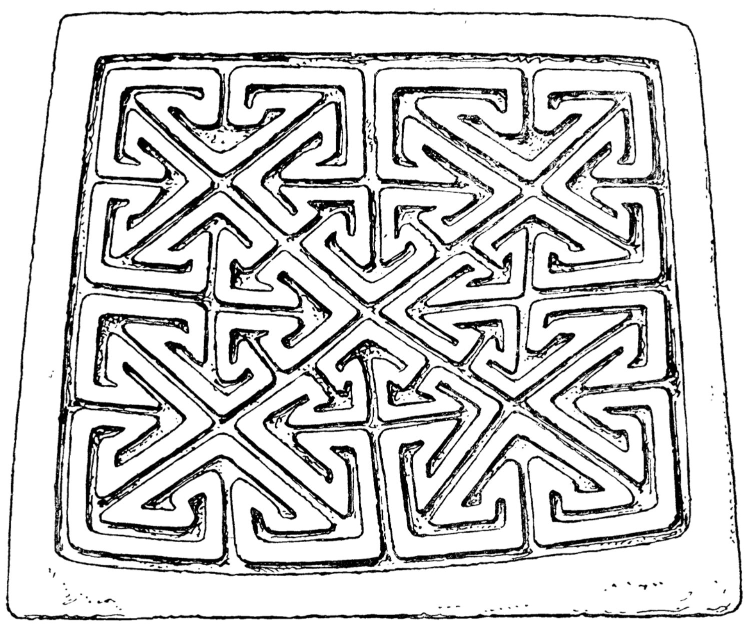
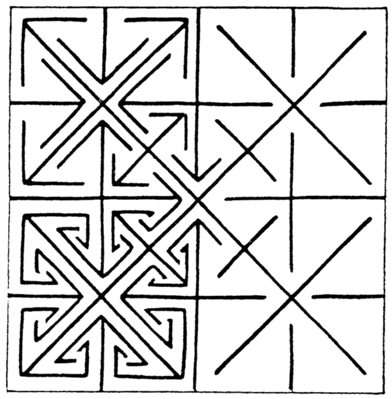


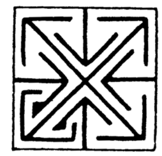
Figure 1: *A Celtic key pattern with self-similarity.*

This is No. 1001 in Romilly Allen's classification (figure 2), and he sees it as four copies of a common pattern (figure 3), with some modification.



No. 1001.—Formed by repeating No. 995 in a double vertical row, with some variations in the centre.
Wales—Penmon, Anglesey.
Ireland—St Brecan's Bed, Aran Mór.

Figure 2: *Romilly Allen's analysis of figure 1.*



No. 995.—A square divided into eight triangles, each filled in with single straight-line spirals.
Berneray.
Invergowrie, No. 1.
St Andrews, No. 7.
,, No. 9.
Wales—
Llantwit Major, Glamorganshire.
Llangevelach, ,,
Ireland—
Termonfechin.
Tuam.

Figure 3: *Romilly Allen's basic unit.*

The Underlying Structure

Like much insular art of this period the image in figure 1 seems designed to dazzle the eye by its intricacy. The immediate impression is of a multitude of arrows, pointing in various directions. Closer inspection reveals that each arrow is produced as the space between other arrows, with perception flipping between them in a way typical of figure/ground illusions. Many well-known fractal curves, such as Koch's snowflake, have the same property: given only a small piece of the curve it is impossible to tell which side is inside the snowflake. This suggests that there is more to this particular key pattern than Romilly Allen's stated analysis implies, and that it could be the basis of a self-similar fractal.

In fact the skeleton of the structure illustrated in figure 2 is equally suggestive. Compare it with figure 4 (taken from Mandelbrot [6]). The "outer" skeleton would correspond exactly with the grey in figure 4, if it were developed further, and the "inner" skeleton with the black. This provides the basis of a recursive program to draw the key pattern, which can then be run to any depth.

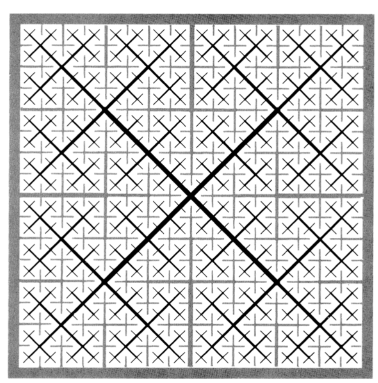


Figure 4: *Mandelbrot's illustration based on Cesàro's triangle sweep.*

A Recursive Program

The program (using turtle geometry) will draw the lines of figure 2, the grooves cut by the mason, rather than the continuous curve that is probably easier to see in figure 1 (another example of interchanging figure and ground).

Cesàro's actual construction replaces each line with a pair of lines at a right-angle, but we need the lines of each skeleton (inner and outer) to stop before they meet the intersections of the other skeleton, so more is involved. We need to distinguish a (full length) line from a short line, and a forward path from a backward path.

The idea is that a full line is replaced by: half line, 90° turn, half short line forwards, 180° turn, half short line backwards, 90° turn, half line. A short line forwards is replaced by: half line, 90° turn, half short line forwards, 180° turn, half short line backwards, 90° turn, half short line forwards. A short line backwards is replaced by: half short line backwards, 90° turn, half short line forwards, 180° turn, half short line backwards, 90° turn, half line. The difference is that a short line forwards has the short bit at the end, a short line backwards has the short bit at the beginning.

The outer skeleton is made by applying the recursion rules to the edges of a square; the inner skeleton by applying the recursion rules to four pairs of back and forward short lines from the centre of the square, with a scale factor of $1/\sqrt{2}$. Figure 5 is the basic skeleton of figure 2; figure 6 takes the recursion one step further.

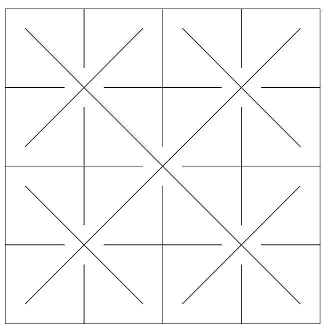


Figure 5: *The skeletons at recursive depth 2.*

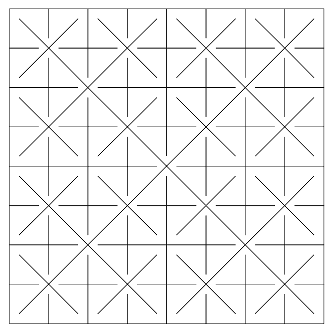


Figure 6: *The skeletons at recursive depth 3.*

Inspection of figure 3 makes it clear that the arrow-heads are different on the outer and inner skeletons. The lengths of their constituent line-segments need considerable adjustment to achieve an aesthetically pleasing effect. Figure 7 shows figure 5 with arrow-heads added.

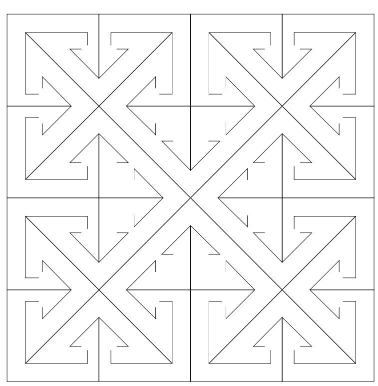


Figure 7: *Arrow-heads added to the basic skeleton of figure 5.*

It is obvious that something is missing because not every arrow appears as the space between other arrows. The clusters of three arrows at the ends of the inner skeleton work properly, but corresponding clusters on the outer skeleton imply extra arrows on the inner skeleton. There is not much space for the additional arrows, so they need to be quite small. The rule is to add a pair of arrows to any full-length line in the inner skeleton.

The clusters of three arrows in the outer skeleton now need special treatment. Usually in the program the actual drawing happens at the bottom level of recursion, and a single line is drawn. Now we need to draw a complete cluster at the first level up, and it consists of three different types of arrow: the central one is smaller, and the side arrows have one side smaller than the other. Again considerable adjustment in the exact lengths is needed to achieve a pleasing impression. Figure 8 shows the recursion taken to level 2, and it corresponds with figures 2 and 5. Figure 9 takes it one stage further, corresponding to figure 6, and demonstrates that the key pattern really can be extended to deeper levels of recursion.

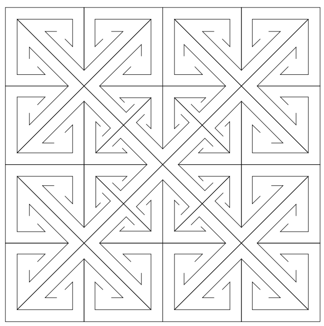


Figure 8: *A recursively generated version of figure 2*

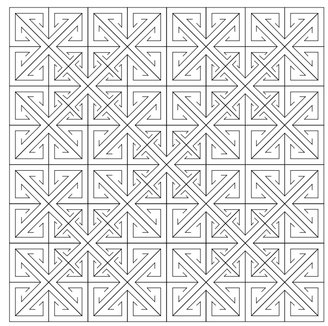


Figure 9: *The next level of recursion.*

The Shape of the Arrows

As Romilly Allen observes (figure 3) these patterns are based on the division of a square into triangles filled with straight-line spirals. It is possible to draw them recursively because an isosceles right-angled triangle can be repeatedly bisected into smaller copies of itself. Since the edges of the triangles are parallel to the lines of the spirals they determine the shapes of the arrows. Those that form the arrow-heads on the outer skeleton start parallel to a hypotenuse so they take a 45° turn initially; those on the inner skeleton begin with a right-angle. The further addition of the small arrows requires yet more types. There would be greater self-similarity if all of the arrows were the same shape.

There is a construction, which also relies on the special properties of isosceles right-angled triangles, that will generate a single arrow shape that fits together in the right way. Figure 10 shows part of the smallest bottom left-hand square of figure 8. ABCDE is half an arrow from the inner skeleton and WXYZ half of one from the outer skeleton. WA is a side of the square (of unit length, say), and WX is a “short line” of the outer skeleton, some fractional length, f, say. AB is a corresponding “short line” of the inner skeleton, of length $\sqrt{2}f$. BC and XY are also corresponding segments so they must also be in the same ratio, $\sqrt{2}:1$. YC is at 45°. Calculation leads to the result that $XY = (3f-1)/2\sqrt{2}$, and $CD = (5f-3)/4$. Figure 11 shows how the two lines spiral around each other, following the edges of a repeatedly bisected isosceles right-angled triangle. Corresponding edges are in the ratio $\sqrt{2}:1$, so the line segments in each spiral path halve at every turn, which is a right-angle, producing a square path that is an approximation to a logarithmic spiral.

The self-similar nature of the spiral ensures that it is no longer necessary to modify it to accommodate the additional small arrows. Figures 12 and 13 show the pattern in figures 8 and 9 with spiral arrow-heads.

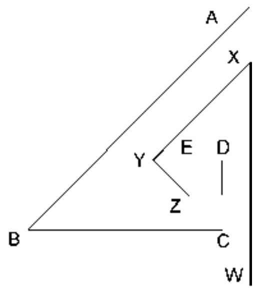


Figure 10: *Parts of two geometrically similar arrows.*

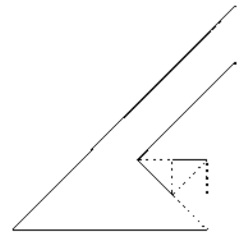


Figure 11: *The relationship between the lines.*

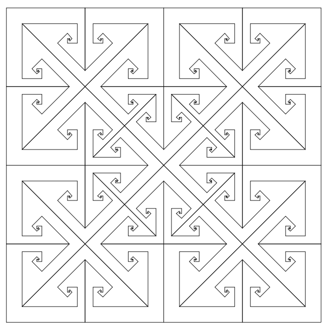


Figure 12: *The pattern with uniform arrow shape.*

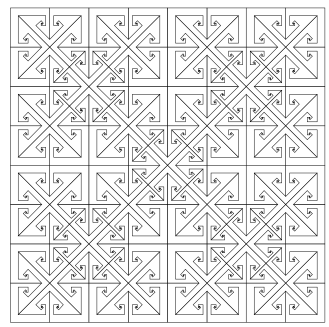


Figure 13: *The next level of recursion.*

Fractals?

Jean-Marc Castera has described an Islamic tiling found in Andalusia that also displays self-similarity [7]. He comments that it suggests the possibility of infinity more strongly than periodic patterns, but does not argue explicitly that this was the intention of its designer. Celtic designs generally lack the coherence and large-scale structure of architectural ornament in the Islamic world, relying more on the simple accumulation of detail for their effect, and if there is an implication of infinity it is of infinite intricacy. Although both designs suggest fractal geometries to a modern, mathematically sophisticated audience, they could not have been seen in this way when they were created. Islamic mathematicians were well in advance of the Christian world at the time, but even they had not developed the modern concept of infinity, or recursion, but the existence of these examples suggests that there is an intuitive sense of recursion that was not formalised until modern times.

Celtic art in general, and key patterns in particular, consists of elaborations of a limited range of standard devices. For example only a few of the possible symmetries are used, even when visual impressions suggest a different one. Romilly Allen, followed by other archaeologists, typically ignores the overall appearance of symmetry, sometimes to the extent of distorting patterns to conform to the standard schemes, keeping only the topological features [8]. His analysis of pattern No. 1001 (figure 2) similarly ignores what seem obvious structural features, probably for the sake of his classification.

The other known appearance of this pattern is in Inishmore, in the Aran Islands, off the west coast of Ireland. Since the only known examples occur on islands, allowing easy communication by sea, it is reasonable to assume that they are related. The Irish example (figure 14) appears on a cross shaft that is part of a large collection of ancient monuments [9]. Another (figure 15) also has a key pattern consisting of four copies of the basic motif (figure 3), with variations in the centre, but to modern eyes it looks like a mistake. It could be a trial version, or an inept copy, but it seems unlikely that the carver would work directly on the stone without planning first. Perhaps, like heraldic quartering, which sometimes produces peculiar results, it has some meaning, now unknown, or maybe it is exactly what the artist intended: he simply had a different aesthetic.

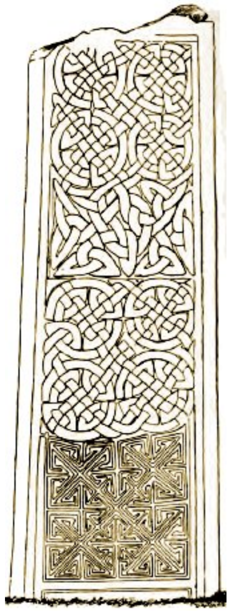


Figure 14: *Cross-shaft at Leaba Brecan, Inishmore.* **Figure 15:** *Another cross-shaft from the same location*

It seems probable that the artist of figure 14, maybe after seeing the pattern in figure 15, had considered the possibility of using clusters of three arrows instead of a single one, then made adjustments until the pattern worked. Having neither the concept of recursion, nor the modern propensity for taking things to extremes, he developed it no further. Another factor could be the limitations of the medium, although the scribes of an earlier period could have used the pattern in figure 9. Pages from books like the Lindisfarne Gospels are at least as detailed, but by this date the age of such masterpieces was coming to an end, and no design with deeper recursion was attempted.

References

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[2]Bain I., *Celtic Key Patterns*, Constable, 1993.

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[4]Cromwell P.R., *The distribution of knot types in Celtic interlaced ornament,* Journal of Mathematics and the Arts, Vol.2 No.2, June 2008.

[5]Nash-Williams V.E., *The Early Christian Monuments of Wales*, University of Wales Press, 1950.

[6]Mandelbrot B.B., *Fractals Form, Chance and Dimension* Freeman, 1977.

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[8]Gailiunas P., *Celtic Key Patterns*, Symmetry: Culture and Science, Vol. 20 Nos. 1-4, 2009.

[9]www.inishmore.antiochian.co.uk/ (accessed December 2009).