# Some Three-dimensional Self-similar Knots

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## Abstract

We present a selection of three-dimensional self-similar knots. The most successful method we have found for creating these is to use a geometric basis consisting of a starting surface in which one or more smaller copies of the surface are embedded. Points along the surfaces are connected by strands outside, between, and inside the surfaces to form a starting knot. Self similarity is introduced by an iterative process in which the contents of the smaller surface(s) are replaced with the contents of the larger surface. A distinction is drawn between thickened two-dimensional knots and true three-dimensional knots, with examples of each type. Self similarity allows for a combination of complexity and order, leading to novel and interesting sculptural forms.

## 1. Introduction

A mathematical knot is a non-intersecting closed curve in space [1]; for the purposes of this paper, three-dimensional space. While it is understood that knots live in 3-space, they are usually depicted in 2-dimensional drawings. These drawings can be both beautiful and fascinating; hence the popularity of knots in the decorative arts. Three-dimensional representations of knots can be difficult to visualize and manipulate, but they can describe graceful and beautiful space curves. They can also possess a feature found in much of the more interesting abstract sculpture in the fact that their appearance can differ dramatically depending on the viewing angle.

In this paper, we describe several three-dimensional knots that are created by iterative processes and as a result possess self similarity. In order to prevent the knots from being excessively complex, only a few iterations are performed. However, the construction is designed to allow an arbitrarily large number of iterations while maintaining a true knot; i.e., a single strand. In the limit of an infinite number of iterations these would be fractal structures. The intent of this work is the creation of novel and interesting sculptural forms and recreational mathematics, rather than contributing new material to the mathematical study of knots.

The line drawings in this paper were created with the commercial drawing programs FreeHand and Illustrator. The drawings of three-dimensional knots were created using KnotPlot [2], with retouching of postscript files performed in Photoshop.

The third and fourth knots described here were produced using a three-dimensional analog of the method described in Reference 3 for creating two-dimensional iterated knots. This method utilizes a starting curve (surface in three dimensions) in which smaller copies of the curve (surface) are embedded. Points on the surfaces are connected by strands outside, between, and inside the surfaces to form a starting knot (see Figure 3a for an example). Self similarity is introduced by an iterative process in which the contents of the smaller surface(s) are replaced with the contents of the larger surface. This step may be carried out as many times as desired. In practice, the knots tend to be come relatively complex after only a few iterations.

The strands are specified by a series of Cartesian coordinates. Simply connecting these leads to a knot that is a series of straight-line segments, such as that shown at top left in Figure 3a. In most of the figures, for example the right side of Figure 1, smooth curves are shown. These are created by KnotPlot using a built-in smoothing function, with the specified Cartesian coordinates acting as control points. KnotPlot also allows real-time manipulation of knots in three dimensions. The KnotPlot files created for this paper are available for downloading [4].

The only previous examples of true three-dimensional self-similar knots of which we are aware have been presented by Séquin [5].

## 2. Pseudo Three-dimensional Knots

We previously described two different methods for designing iterated knots and presented several examples [3, 6]. Additional examples may be seen online [4]. Some of these knots lend themselves well to the creation of a three-dimensional version by basically thickening the knot. We consider such knots to be pseudo three dimensional rather than true three-dimensional knots. Such thickened two-dimensional knots are of limited interest because the whole story is really told by a particular viewing angle. Additional views, while they can look quite different, don't reward the viewer with any great surprises. However, thickened self-similar knots can still possess considerable esthetic appeal.

Figure 1 shows a thickened version of a two-dimensional self-similar knot described previously [3]. The geometric basis for the knot is two 1/3 scale rectangles placed within a starting rectangle. The linear scaling factor between successive generations is $1/\sqrt{3}$ ($\approx 0.577$). The number of crossings added doubles with each iteration.

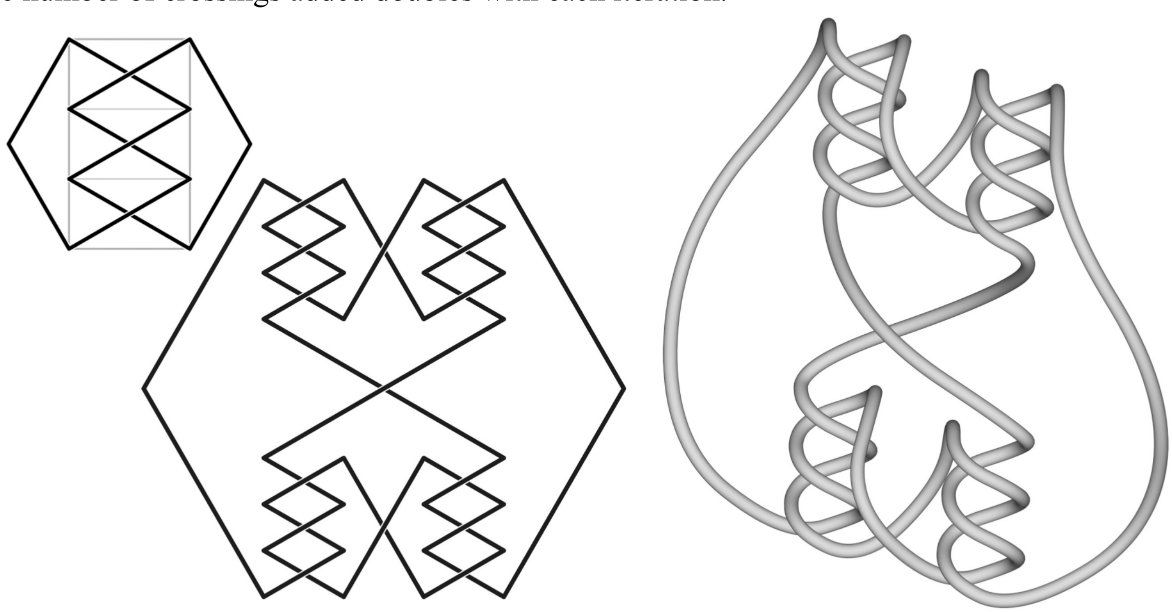

**Figure 1**: *A line drawing showing a two-dimensional projection of an iterated knot, with the geometric basis and starting knot at upper left and knot after two iterations at center. A particular view of a three-dimensional version created in KnotPlot is shown at right.*

## 3. True Three-dimensional Knots with One Singularity

A singularity is a point about which things become infinitely dense or crowded. In the case of a fractal knot in 3-space, it would be a point about which any finite sphere would contain an infinite number of crossings. When constructing a knot with one singularity, each iteration would add one smaller copy of a set of strands. In contrast, a more complicated construction could be employed in which each iteration would add twice or more times as many copies of a set of strands as were added in the previous iteration. Knots with only one singularity are less complex and in general easier to handle and easier for the viewer to take in. We present two such examples in this section.

The first example utilizes double helices similar to the four seen in Figure 1 at right, with three crossings each. The starting knot is created by arranging two of these double helices at right angles and by joining the ends of the strands to form a single strand (upper left in Figure 2, shown from two different angles). The knot is iterated by replacing the middle third of each double helix with a scaled-down three-crossing double helix oriented at a right angle to the larger double helix. The scaling factor between successive iterations is $1/\sqrt{3}$ ($\approx 0.577$). As a result of the fact that there are double helices in only two of the three orthogonal directions in 3-space, the exact configuration of the smaller double helices differs from one iteration to the next, alternating between two configurations. Every two iterations, then, a characteristic group of strands repeats in the center of the knot at 1/3 scale, as seen in Figure 2.

The second example, shown in Figures 3 and 4, is based on the trefoil knot. The scaling factor from one iteration to the next is 0.75 for the final version of the knot. It is generally desirable to iterate with as large a scaling factor as possible to prevent features from becoming excessively small too quickly. The initial starting knot, shown in Figure 3a, had a scaling factor of 0.5. Refining the design resulted in the scaling factor increasing to 0.75, allowing more iterations to be shown at once (Figures 3c and 4).

The geometric basis is two nested cubes, as shown at left in Figure 3a. The knot is specified in KnotPlot as an ordered set of coordinate points, indicated by balls in the figure. The poly-line connecting these control points in sequence specifies the knot in the form of a balls-and-sticks knot. To increase the esthetic appeal, this poly-line is turned into a smooth spline curve, defined by the control points, as shown in Figure 3a at right for the starting knot. However, care has to be taken that the smoothing process doesn't alter the topology of the configuration and thus the knottedness of the curve. In the starting knot, which has nine crossings, the strand in each of three lobes goes from the outer cube to the inner cube in three straight-line segments, as shown in Figure 3a by a heavy black line. The inner cube is met on the opposite side from the outer cube. Iterating this, another three straight-line segments brings the strand back to the original side, as shown by a heavy black line in Figure 3b. It is seen that the strand in each lobe traces out one full loop of a spiral in two iterations. After the first iteration, the number of crossings appears to be 15 [7], so that six crossings are added with each iteration.

After a few iterations, the knot basically has the form of three orthogonal spirals that penetrate one another. In order to add more visual interest, the spirals were modified to change in depth as they wind inward or outward, as opposed to being planar. The last view in Figure 4 clearly shows this feature.

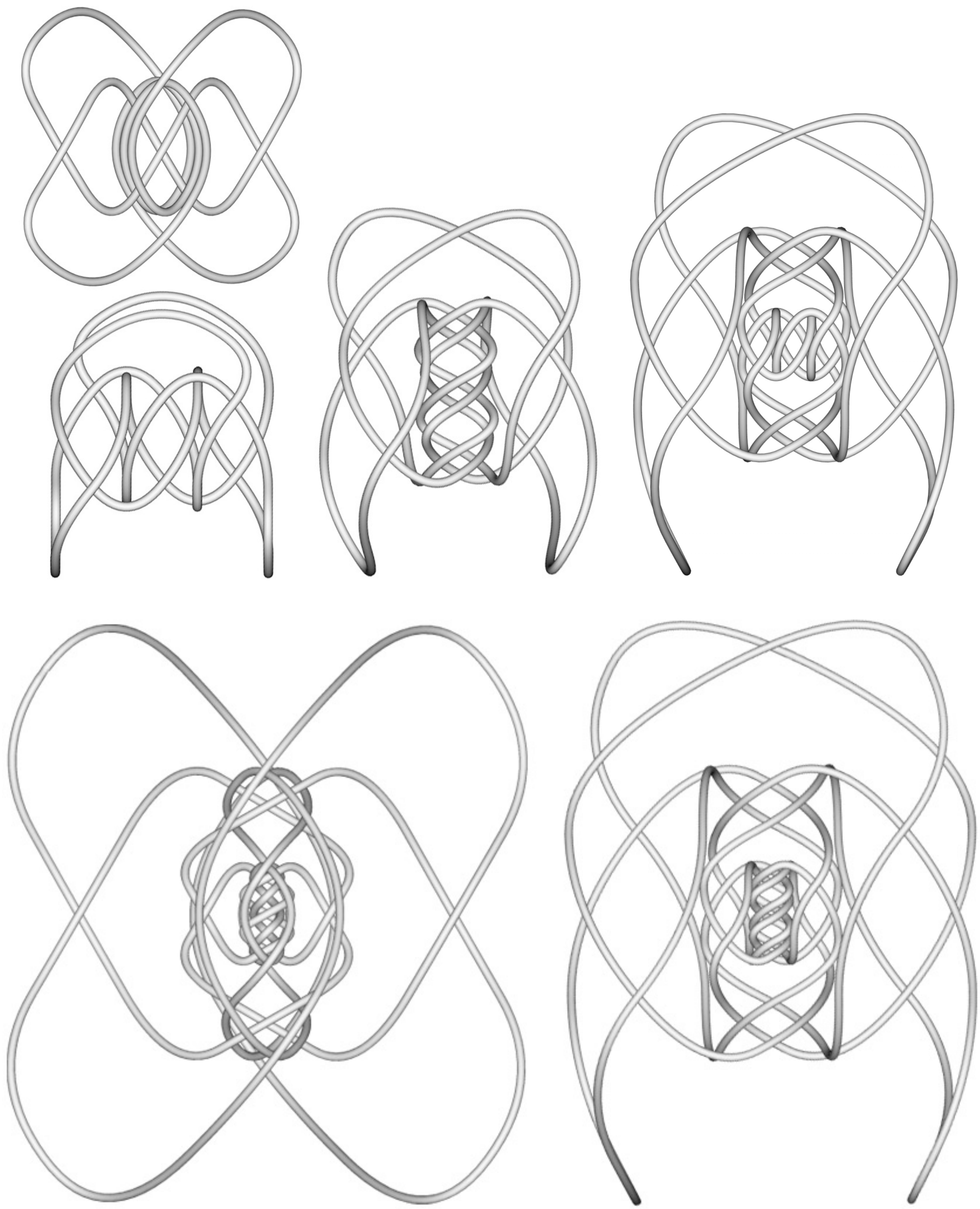

**Figure 2**: A self-similar knot with one singularity and a scaling factor of 1/√3. The starting knot is shown at top left from two different angles, the knot after one iteration at top middle, and after two iterations at top right. The bottom two figures show the knot after three iterations, from two different angles.

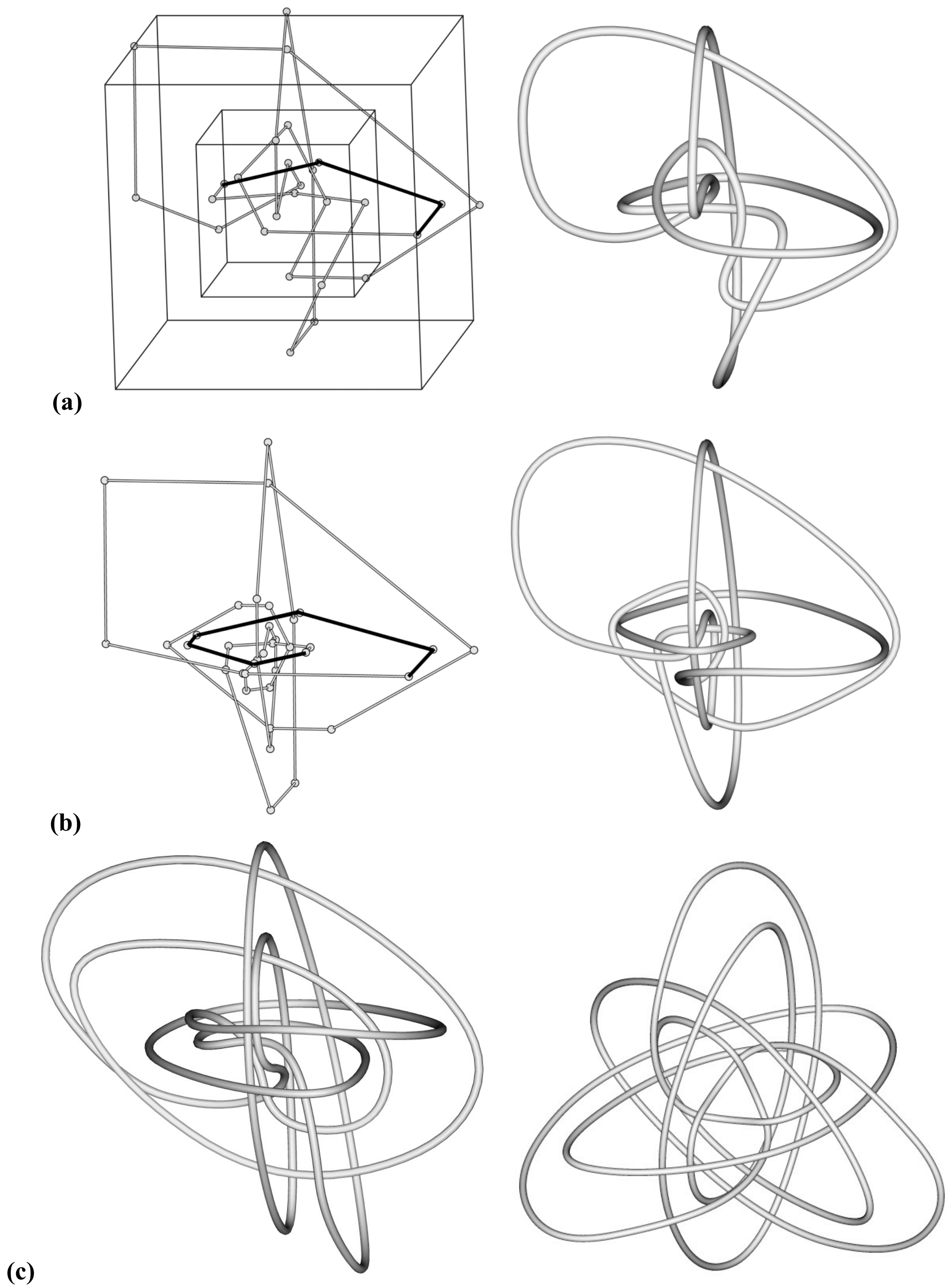


**Figure 3:** *A self-similar knot based on a trefoil knot. a) The balls-and-sticks version of the starting knot is shown at left along with the cubes that form the geometric basis. Each of three lobes goes from the outer to inner cube in three straight-line segments, one of which is shown in black. The smoothed version is shown at right. b) After one iteration, three more straight-line segments form a full loop of a spiral. c) The smoothed knot after two iteration, seen from two different angles. Compared to Figures 3a and 3b, this version has been modified to shrink less rapidly.*

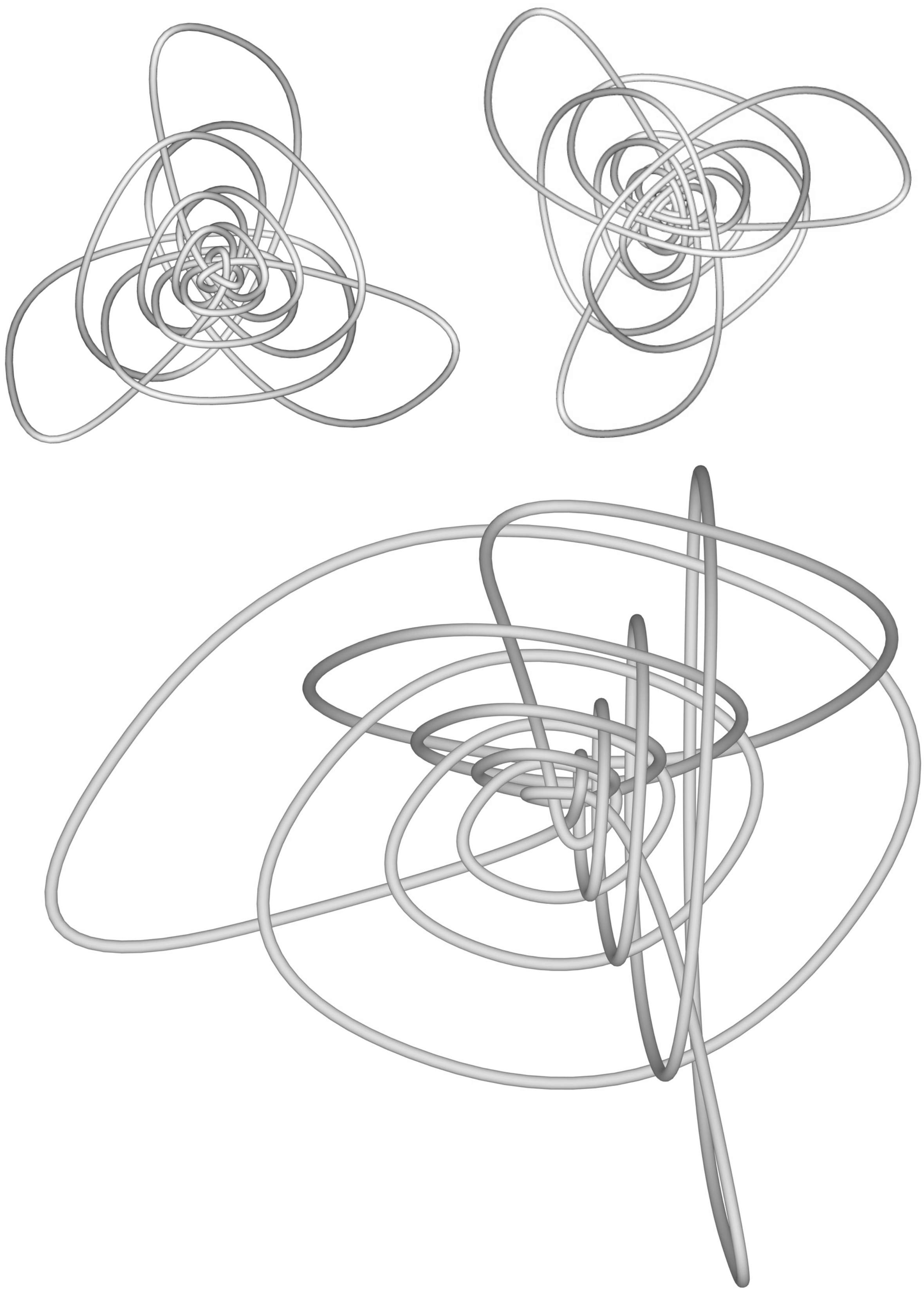

**Figure 4:** *The knot of Figure 3, after six iterations, seen from three different angles.*

## 4. A Knot with a More Complicated Geometric Basis

In this section, we present an example of a knot in which the number of smaller copies of a set of strands multiplies by a factor of six with each generation. The geometric basis for the knot is a group of six octahedra nested in a larger octahedron, as seen at top right in Figure 5. Strands run along diagonals and edges of the octahedra, and the scaling factor between successive generations is 1/3.

The singly-iterated knot can be thought of as a compound knot consisting of six trefoil knots distributed along a path that itself forms a trefoil knot. The second iteration can be similarly thought of as a compound knot of compound knots. Note that this knot does not have a finite number of attractor points that is fixed from one iteration to the next. Rather it is more akin to a space-filling curve, getting uniformly dense in many areas. While this sort of knot may be more interesting mathematically, the knots above, which possess a single singularity at their centers, are more effective as sculptural forms in our opinion.

## 5. Conclusion

We have presented a selection of three-dimensional self-similar knots. Such knots constitute novel and interesting sculptural forms. Self similarity allows for a combination of complexity and order. The design of true three-dimensional knots is challenging and generally requires several attempts to arrive at a successful solution, with several additional refinements required to create an esthetically pleasing knot. After two or more iterations, the strands must be kept narrow in order to leave space between them in the densest areas. For this reason, varying the strand width would in some cases improve the appearance of the knots. Unfortunately, variable strand width is not allowed in the current version of KnotPlot. Full appreciation of these structures is only possible by three-dimensional manipulation, as the appearance varies dramatically depending on the viewing angle.

## References


[1] Colin C. Adams, *The Knot Book,* The American Mathematical Society, Providence, RI, 2004.

[2] KnotPlot was written by Rob Scherein and is available at http://www.knotplot.com/download/.

[3] Robert Fathauer, *A New Method for Designing Iterated Knots,* in Proceedings of the Bridges 2009 Conference, edited by Craig S. Kaplan and Reza Sarhangi, pp. 251-258, 2009.

[4] Robert Fathauer, web site entitled *Compendium of Fractal Knots*, http://www.members.cox.net/knotcompendium/index.html.

[5] Carlo Séquin. *Tangles Knots,* in Proc. Art+Math=X Intl. Conf., edited by Carla Farsi, pp. 161-165, 2005.

[6] Robert Fathauer, *Fractal Knots Created by Iterative Substitution,* in Proceedings of the Bridges 2007 Conference, edited by Reza Sarhangi and Javier Barrallo, pp. 335-342, 2007.

[7] In KnotPlot, the balls and sticks can be moved, allowing the knot to be simplified. This in principle allows one to determine the minimum number of crossings. However, for a relatively large number of crossings (roughly greater than a dozen), it is difficult to be sure that further simplification isn't possible.

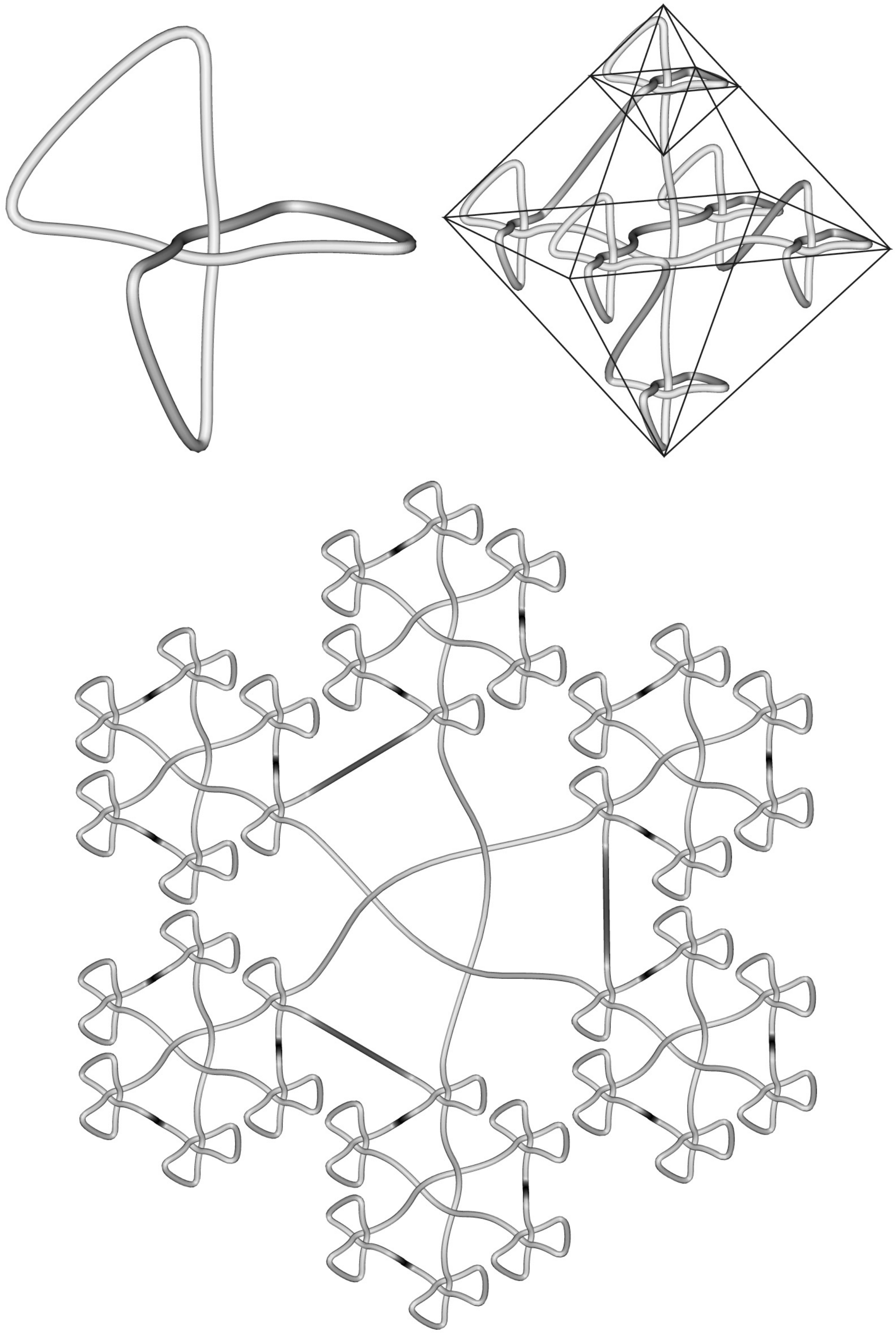

**Figure 5:** *A self-similar knot based on a trefoil knot that incorporates six smaller copies with each iteration. The starting knot is shown at top left, after one iteration at top right, and after two iterations at bottom. At top right, the large octahedron and one of the six smaller octahedra that form the geometric basis are shown,*