Tiling the Musical Canon with Augmentations of the Ashanti Rhythmic Pattern

Carl Bracken[1], Gary Fitzpatrick[2], Nadya Markin[3]
[1]Department of Mathematics
National University of Ireland Maynooth
Co. Kildare, Ireland
Email:carlbracken@yahoo.com
[2]Email:garyfitz@dublin.ie
[3]Claude Shannon Institute
School of Mathematical Sciences
University College Dublin
Bellfield, Dublin 4, Ireland
Email:nadyaomarkin@gmail.com

Abstract

We discuss the problem of constructing a tiling of the musical time-line with a number of instruments (called voices) all of which are playing according to variations of a particular rhythmic pattern. We show that the Ashanti rhythmic pattern allows a tiling in six voices.

1 Introduction

The term rhythmic canon was coined by the composer Olivier Messiaen (1908-1992). The rhythmic canon dictates when each instrument in a composition may play a note or be silent. Each instrument should play the same rhythm but start at a different time. If the rhythmic canon is such that at every time interval exactly one instrument can be heard, then the canon is said to be tiled. Messiaen himself referred to the sound of a rhythmic canon as "organised chaos" [5]. In his composition *Harawi*, Messiaen uses a three voice canon with each voice playing according to the following rhythmic pattern.

100100001000000010000100100010010000001001000101010100100000100

A 1 indicates that the instrument plays a note, while a 0 means it is silent. When the three voices play together they play with the same rhythm but start at different times to give the following rhythmic canon.

V_1 : 100100001000000010000100100010010000001001000101010100100000100
V_2 : 00100100001000000010000100100010010000001001000101010100100000100
V_3 : 0000100100001000000010000100100010010000001001000101010100100000100

This canon has the property that there is an instrument playing on almost every beat and with just a few exceptions there is only one instrument playing, so it is almost perfectly tiled. In this article we are concerned with perfectly tiled canons, i.e., canons in which there is one and only one instrument playing at every time interval. Today there are many musicians who use tiled canons in their compositions. In particular we would refer the reader to the work of Tom Johnson [4].

2 Polynomial Representation of the Rhythmic Canon

The binary representation of rhythmic patterns used above is useful to anyone wanting to analyse a pattern or compose a piece of music based on the described rhythm. However, if we wish to construct a rhythmic canon with particular properties or check whether a given set of patterns tile the musical time-line, then polynomial representation is more useful.

The polynomial that represents a given rhythmic pattern is simply a polynomial in x with integer coefficients 0 or 1. If in the binary representation the pattern has a 1 in the i^{th} position, then the coefficient of x^i is 1, otherwise it is zero. Note that for a rhythmic pattern of period n (i.e., one that repeats every n beats) we consider the binary representation as starting at position 0 and finishing at position $n-1$. This means the binary pattern of Messiaen

100100001000000010000100100010010000001001000101010100100000100

will be written as

$$1+x^3+x^8+x^{16}+x^{21}+x^{24}+x^{28}+x^{31}+x^{38}+x^{41}+x^{45}+x^{47}+x^{49}+x^{51}+x^{54}+x^{59}.$$

We shall refer to polynomials which have 0 and 1 as coefficients as $0-1$ polynomials. As each rhythmic pattern has some period n, the powers in its polynomial representation are reduced modulo n. Therefore its corresponding polynomial can be regarded as an element of the ring $\mathbb{Z}[x]/(x^n-1)$. Multiplication of a polynomial by x^i will shift the rhythm by i positions, meaning it will start i beats later. Let $P(x)$ be a $0-1$ polynomial that describes a rhythm to be used in a canon. If there exists another $0-1$ polynomial say $Q(x)$ such that $Q(x)P(x) = 1+x+x^2+\ldots+x^{n-1}$, then the canon can be tiled with the pattern $P(x)$. We refer to $P(x)$ as the inner rhythm and the polynomial $Q(x)$ is called the outer rhythm. Each voice plays according to inner rhythm, while the outer rhythm determines when each voice starts.

3 Examples of Tiled Canons

3.1 A Simple Tiling

Consider the rhythmic pattern of period 12 described by the polynomial $P(x) = 1+x^5+x^7$. We choose the outer rhythm $Q(x)$ to be $1+x^3+x^6+x^9$. As the period is 12, all powers of x will be reduced modulo 12. It can be easily verified that

$$Q(x)P(x) = (1+x^3+x^6+x^9)(1+x^5+x^7) = 1+x^2+x^3+x^4+\ldots+x^{11},$$

hence the time-line is tiled with four voices, one for each term of $Q(x)$. The rhythmic patterns are periodic, so in the binary representation we may write the following canon.

V_1 : 100001010000100001010000100001010000
V_2 : 000100001010000100001010000100001010
V_3 : 010000100001010000100001010000100001
V_4 : 001010000100001010000100001010000100

Here each voice repeats the pattern three times. It can be seen from above we have a perfect tiling of the time-line.

As $Q(x)P(x) = P(x)Q(x)$, we can obtain another tiling by interchanging the roles of the outer and inner rhythms. That is, we can obtain the following canon in three voices.

$$V_1 : 100100100100100100100100100100100100$$
$$V_2 : 010010010010010010010010010010010010$$
$$V_3 : 001001001001001001001001001001001001$$

3.2 Vuza Canons

In the above example we see that the rhythm $1 + x^3 + x^6 + x^9$ has a smaller period than 12. It repeats every three beats. If a polynomial is invariant under multiplication by x^k, then the polynomial's rhythm has period k. If a tiling has no period smaller than n in either the inner or outer rhythms then it is said to be of *maximal category*. It was shown by Vuza [6] that there exists no tiled canon of maximal category with period less than 72. He also provided an algorithm for producing canons of maximal category of period 72 and 120. Much work has been done by both mathematicians and musicians on canons of maximal category, or as they are now called Vuza canons. One example of a Vuza canon is given by polynomials

$$P(x) = 1 + x + x^5 + x^6 + x^{12} + x^{25} + x^{29} + x^{36} + x^{42} + x^{48} + x^{49} + x^{53},$$

$$Q(x) = 1 + x^8 + x^{18} + x^{26} + x^{40} + x^{58}.$$

As mentioned earlier this will allow for two tilings of the time-line. For more on this topic we refer the reader to [1] and references there in.

3.3 Augmented canons

Another class of tiled rhythmic canons are the augmented canons. In an augmented canon different voices will still play the same rhythmic pattern but some may be stretched by a factor r so that they take longer to complete their cycles, while the non-stretched voices will be repeated r times so that the canon can be tiled. In terms of polynomials, an augmentation of $P(x)$ by a factor of r corresponds to the polynomial $P(x^r)$.

Let k be the period of a rhythm represented by $P(x)$. Suppose we wish to tile the canon with two versions of the same rhythm say $P(x)$ and $P(x^r)$. Note that the augmented pattern $P(x^r)$ is periodic in kr. We can fill kr time intervals with r copies of the pattern corresponding to $P(x)$. In polynomial notation the r copies are created by the polynomial

$$(1 + x^k + x^{2k} + x^{3k} + ... + x^{(r-1)k}).$$

Now, to complete a tiling we would have to find two outer rhythms $Q_1(x)$ and $Q_2(x)$ such that

$$Q_1(x)(1 + x^k + x^{2k} + x^{3k} + ... + x^{(r-1)k})P(x) + Q_2(x)P(x^r) = 1 + x + x^2 + ... + x^{rk-1}.$$

The polynomials $Q_1(x)$ and $Q_2(x)$ determine when the voices $P(x)$ and $P(x^r)$ start. The number of terms in $Q_1(x)$ will be the number of voices playing the faster rhythm $P(x)$, while the number of terms in $Q_2(x)$ will be the number of voices playing the slower rhythm $P(x^r)$. We could, of course, try to tile the canon with many different augmentations of the same rhythm, but for the purpose of this article we will consider just one augmentation along with the original pattern.

3.4 Augmented Canons from Traditional Patterns

The purpose of this collaboration (between two mathematicians and a musician) was to try to tile the time line with augmentations of an existing traditional rhythmic pattern and to compose a piece of music on this tiling. Of the large number of (mathematically) possible rhythm patterns that could be used for a given rhythm period only a small number of these are used in practice [3]. It is believed that a rhythmic pattern that has been used for generations probably has some aesthetic quality that a pattern chosen just to satisfy an equation does not. Using a list of traditional African patterns (from [2]), for each period 12 pattern on the list we attempted to solve the equation

$$Q_1(x)(1+x^{12}+x^{24}+\dots+x^{(r-1)12})P(x)+Q_2(x)P(x^r)=1+x+x^2+\dots+x^{12r-1},$$

for some small value of r. That is we had to find $0-1$ polynomials $Q_1(x)$ and $Q_2(x)$ such that the above equation holds. We were able to solve this equation for just one of the patterns. In the next section we show how we derived the solution and demonstrate the resulting tiling.

4 The Ashanti Rhythmic Pattern

4.1 Background

The Ashanti people make up 14% of the population of modern day Ghana. The basic rhythmic pattern behind much of their traditional music is the following period 12 pattern with four onsets

100101001000.

As a polynomial we would write this as

$$P(x)=1+x^3+x^5+x^8.$$

Before attempting to tile with a polynomial it is a good idea to note all the possible differences of the powers of x. This will rule out some shifts by revealing which shifts induce overlap. For example, two of the powers that occur in the Ashanti polynomial are 5 and 3, which have a difference of 2. So a shift by 2 places will cause an overlap. The differences are all calculated modulo 12 and no shift greater than 6 need be considered, as this is just a smaller shift in the other direction. The set of all differences of the powers in the Ashanti polynomial is $\{2,\ 3,\ 4,\ 5\}$, so the only possible shifts that can be used in a tiling are 1 and 6. This means that $Q_1(x)$ has to be of the form $x^i(1+x)$ or $x^j(1+x^6)$.

4.2 An Augmented Tiling

To tile the Ashanti pattern with augmentations by a factor of two we are required to find $0-1$ polynomials $Q_1(x)$ and $Q_2(x)$ such that,

$$Q_1(x)(1+x^{12})P(x)+Q_2(x)P(x^2)=1+x+x^2+\dots+x^{23}.$$

Using the restriction derived above we may assume that $Q_1(x)$ is either $1+x$ or $1+x^6$. From here we can rule out the remaining possibilities for $Q_2(X)$ by hand without too much difficulty (or with ease on a computer). We concluded that the above equation has no solutions for $Q_1(x)$ and $Q_2(x)$ that are $0-1$ polynomials, therefore a tiling with the augmentation by two is not possible. We can similarly rule out augmentations by a factor of three.

We proceed to look for a tiling with the Ashanti pattern using augmentations by a factor of four. As before we are required to find polynomials Q_1 and Q_2 such that,

$$Q_1(x)(1+x^{12}+x^{24}+x^{36})P(x)+Q_2(x)P(x^4)=1+x+x^2+\ldots+x^{47}.$$

Again, we know that $Q_1(x)$ is either $1+x$ or $1+x^6$. We found a solution for $Q_1(x)=1+x$. What follows is a summary of the computations that lead to the finding of a viable $Q_2(x)$.

Assuming $Q_1(x)=1+x$ we must find $Q_2(x)$ such that

$$(1+x)(1+x^{12}+x^{24}+x^{36})(1+x^3+x^5+x^8)+Q_2(x)(1+x^{12}+x^{20}+x^{32})$$
$$=1+x+x^2+\ldots+x^{47}.$$

This implies

$$(1+x^{12}+x^{24}+x^{36})(1+x+x^3+x^4+x^5+x^6+x^8+x^9)+Q_2(x)(1+x^{12}+x^{20}+x^{32})$$
$$=(1+x^{12}+x^{24}+x^{36})(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}).$$

A simple rearrangement yields

$$Q_2(x)(1+x^{12}+x^{20}+x^{32})=(1+x^{12}+x^{24}+x^{36})x^2(1+x^5+x^8+x^9),$$

while factoring $(1+x^{12}+x^{20}+x^{32})$ will give

$$Q_2(x)(1+x^{12})(1+x^{20})=(1+x^{12}+x^{24}+x^{36})x^2(1+x^5+x^8+x^9).$$

To facilitate finding a $Q_2(x)$ that would obey the above expression we assume that $Q_2(x)$ has the form $Q_2(x)=x^2(1+x^{24})R(x)$. The equation above turns into the following equation in $R(x)$

$$x^2(1+x^{24})R(x)(1+x^{12})(1+x^{20})=(1+x^{12}+x^{24}+x^{36})x^2(1+x^5+x^8+x^9).$$

It implies

$$(1+x^{24})(1+x^{12})(1+x^{20})R(x)=(1+x^{24})(1+x^{12})(1+x^5+x^8+x^9).$$

Note, we are careful not to say that we divide across by $(1+x^{24})(1+x^{12})$ as the elements $(1+x^{24})$ and $(1+x^{12})$ are not invertible in $\mathbb{Z}[x]/(x^{48}-1)$.

Since all the powers are reduced modulo 48, the polynomial $1+x^{12}+x^{24}+x^{36}=(1+x^{24})(1+x^{12})$ is invariant under multiplication by x^{12}. Therefore we can replace $1+x^5+x^8+x^9$ with $1+x^{41}+x^{20}+x^{21}=(1+x^{20})(1+x^{21})$, since the powers in these polynomials are the same modulo 12. We may now write

$$(1+x^{24})(1+x^{12})(1+x^{20})R(x)=(1+x^{24})(1+x^{12})(1+x^{20})(1+x^{21}).$$

The polynomial $R(x)=1+x^{21}$ satisfies the equation. This gives us the solution

$$Q_2(x)=x^2(1+x^{24})(1+x^{21})=x^2+x^{23}+x^{26}+x^{47}$$

and we now have a tiling.

In the binary notation we may write this tiling as follows.

V_1 : 100101001000100101001000100101001000100101001000
V_2 : 010010100100010010100100010010100100010010100100
V_3 : 001000000000001000000010000000000010000000000000
V_4 : 000000010000000000000001000000000001000000010000
V_5 : 000000000010000000000000001000000000001000000010
V_6 : 000000000001000000010000000000010000000000000001

What follows is an excerpt from a piece of music composed on the above tiling. We can see from this section of the score that all the instruments have been introduced and that at every beat one and no more than one instrument is playing.

References

[1] M. Andreatta, C. Agon, E. Amiot, "Tiling problems in music composition: Theory and Implementation", *Proceedings of the ICMC, Goteborg*, pp. 156-163, (2002).

[2] E. Thul, G. Toussaint, "A comparitive phylogenetic-tree analysis of African timelines and North Indian talas ", *Proceedings of the* 11^{th} *Bridges Conference on Mathematical Connections in Art Music and Science*, pp. 187-194, (2008).

[3] G. Toussaint, "Mathematical features for recognizing preference in Sub-saharan African traditional rhythm timelines", *LNCS, Pattern Recognition and Data Mining, 3686*, pp. 18-27, (2005).

[4] Tom Johnson, Wikipedia pages on Tom Johnson, $http://en.wikipedia.org/wiki/Tom_Johnson_(composer)$.

[5] O. Messiaen, "Traité de rhythme, de couler, et d'ornithologie", *Editions mysicales Alphonse Leduc*, Paris, (1992).

[6] D.T. Vuza, "Supplementary sets and regular complementary unending canons", *Perspectives of New Music*, Nos. 29(2) pp. 22-49; 30(1) pp.184-207; 30(2) pp.102-125; 31(1), pp. 270-305, (1991).