Non-Flat Tilings with Flat Tiles

Rinus Roelofs
Sculptor
Lansinkweg 28
7553AL Hengelo
The Netherlands
E-mail: rinus@rinusroelofs.nl
www.rinusroelofs.nl

Abstract

In general a tiling is considered to be a set of tiles placed next to each other in a flat plane. The tiles are placed in the plane in such a way that there are no gaps and no overlaps. But what if we leave out the condition that the plane has to be flat? For when there are no gaps and no overlaps between the tiles we still can call it a tiling. The consequences for the possible shapes of the tiles in non-flat tilings as well as the possible symmetrical structures that can be used are discussed in this paper.

1. Tilings

1.1. Definition. A tiling, or tessellation, is a covering of a plane without gaps or overlaps by polygons, all of which are the same size and shape. That is one of the definitions of a tiling. Another definition is the following: Tiling: a pattern made of identical shapes; the shapes must fit together without any gaps and the shapes should not overlap. Although the second definition doesn't speak about a plane, it is mostly assumed that the tiles do cover a plane. But we can take the definition literal, and then the only conditions are that the tiles do not overlap and do not leave gaps. Tilings in which all the tiles have the same shape are mostly called monohedral tilings [1]. In this paper all the tilings will be monohedral. In the example of Figure 1 we see a set of identical L-shaped tiles. With these tiles we can make a tiling as a covering of a plane. And there are several ways to do this.

Figure 1: *L-shaped tiles.*

Figure 2: *A non-flat tiling.*

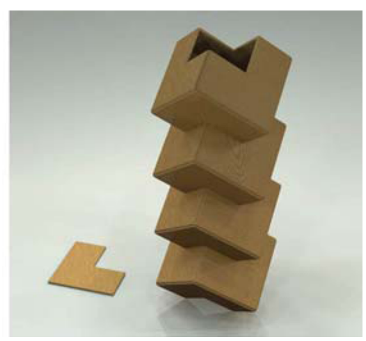


Figure 3: *Cylindrical tiling.*

1.2. Non-flat tilings. Besides the possibility shown in Figure 1, where the tiles are put together in such a way that they cover a flat plane, there are a few more ways to put the tiles together under the condition that we do not want overlaps or gaps in the construction. In Figure 2 the L-shaped tiles are set up in such a way that we can make a three dimensional construction. As you can see the structure is made out of identical shapes. The tiles do not overlap and there are no gaps in the structure. To call it a tiling in the traditional way, the only remark one can have is that it is not a covering of a flat plane. But it still is a tiling. Also when we combine the tiles in the way shown in Figure 3, the resulting structure is a tiling.

1.3. Squares. There are more shapes that can be used to create non-flat tilings and some of them are well known. The square can be used to tile a flat plane, but we can also make non-flat tilings using the squares (Figure 4), again there are no overlaps and no gaps in the structure. The most well known way to combine the squares in a three-dimensional way is the cube (Figure 5), although in general we do not call this a tiling.

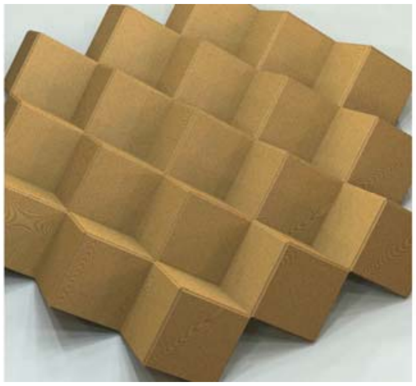


Figure 4: *Non-flat tiling with square tiles.*

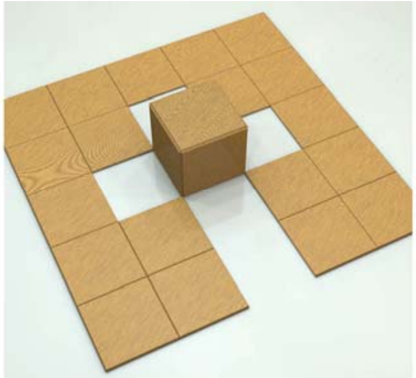


Figure 5: *Non-flat tiling: the cube.*

1.4. Polyhedra. Just as the cube also the other Platonic solids can be interpreted as non-flat tilings (Figure 6). In fact the standard term for “non-flat tiling” is “polyhedron”. And because all the tilings in this paper are monohedral we can call these tilings “monohedra”. So in non-flat tilings the regular pentagon can be used as a tile, which is not possible in the plane.

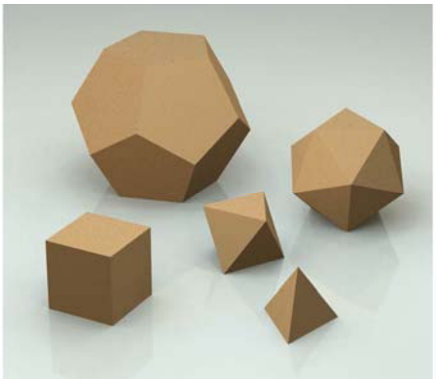


Figure 6: *Platonic solids.*

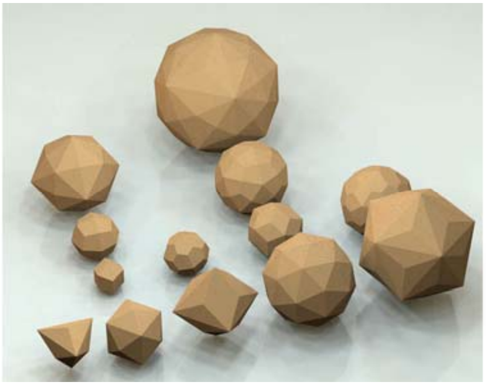


Figure 7: *Platonic solids.*

1.5. Duals of the Archimedean Solids. The Archimedean solids all have two or more different faces, or tiles. For a monohedral tiling the pattern has to be made with identical shapes, so therefore the Archimedean solids can not be seen as monohedral tilings. But when we look at the dual of an Archimedean solid then we see that such an object is built with all identical shapes. Figure 7 shows the complete collection of the thirteen duals. Seven of them are built with triangle-shape tiles, in four duals you will find quadrangle tiles and in two cases pentagonal tiles are used. And because each dual is made out of identical shapes, they can be seen as non-flat tilings.

2. Elevation

2.1. Leonardo da Vinci. In the illustrations that Leonardo da Vinci made for Luca Pacioli's book "La Divina Proportione" [2], we can find a remarkable step towards non-flat tiling. This step is called Elevation and Leonardo da Vinci applied this on almost every regular and semi-regular polyhedron. What we mean with Elevation can be comprehended when we look at Figure's 8, 9 and 10.

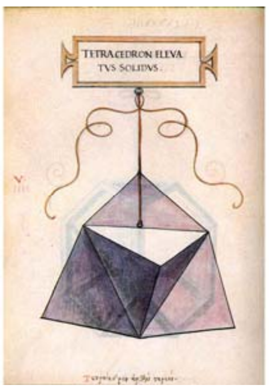


Figure 8: *Tetrahedron elevated.*

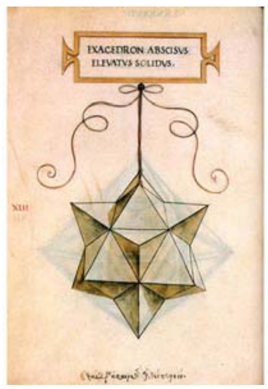


Figure 9: *Cuboctahedron elevated.*

Figure 10: *Rhombic cuboctahedron elevated.*

2.2. The Elevation Process. The process of elevation of a polyhedron can be described as follows: from a face of the polyhedron we take the midpoint and we move this point away from the centre of the polyhedron until the distance between this point and the corner points of the face equals the length of the side of the face. After that we draw a line between the elevated point and each of the corner points of the face. When we do this for each face of the polyhedron the result will be the elevation of the polyhedron.
In Figure 8 we can see the result of this process when we start with the tetrahedron. Because the new faces are all equilateral triangles it is clear that this process only works on triangular, quadrangular and pentagonal faces. In Leonardo's drawings we can find eight of the fourteen possible elevations of regular and semiregular polyhedra. Two of the elevated Archimedean solids are shown in Figure 9 and 10.

2.3. The Elevation of flat tilings. The process of elevation that Leonardo used to create a new set of monohedra can also be applied on flat tilings in which triangles, squares or pentagons are used. In Figures 11 and 12 we can see the non-flat tilings that we will get when we apply the elevation process on the regular tilings with respectively triangles and squares. The resulting tiling is always a non-flat tiling in which only one type of tile is used, which is the equilateral triangle. The possibilities to make non-flat tilings with this tile, the equilateral triangle, are overwhelming. Therefore in this paper we will limit ourselves mainly to cylindrical shapes. When we make a small change with some of the tiles in the tiling of Figure 12, we get the tiling shown in Figure 13. Because this structure is bendable we can transform it into a cylindrical shape (Figure 15). Because all the tiles in these examples areequilateral triangles the tilings belong to the group of "deltahedra".



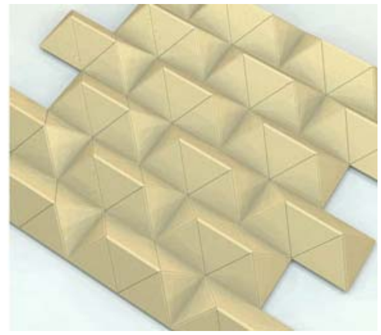


Figure 11: *Eelevation of 3,3,3,3,3,3.* **Figure 12**: *Squares elevated.* **Figure 13**: *Transformation.*

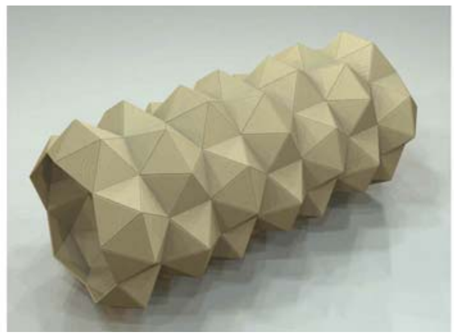
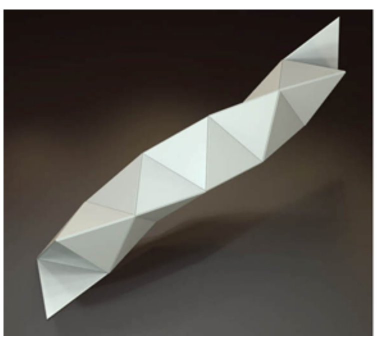



Figure 14: *Bending.* **Figure 15**: *Cylindrical tiling with equilateral triangles.*

2.4. Cylinders – folding/unfolding. The method used in Figures 13 to 15 is just one way of creating cylindrical non-flat tilings with equilateral triangles.



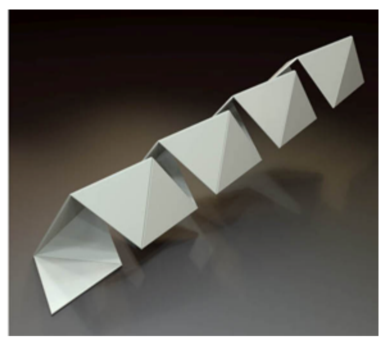


Figure 16: *Tetrahelix.* **Figure 17**: *Unfolding: plane.* **Figure 18**: *Unfolding: strip.*

In Figure 16 we see the tetrahelix, one of the most simplest cylindrical non-flat tiling with equilateral triangles. The tetrahelix can be fold from the normal flat tiling with triangles (Figure 17), but we can also start with a strip of triangles. Just spiralling around as in Figure 18, the strip of triangles will become a tetrahelix. A lot of interesting new shapes can be made by using this way of building non-flat tilings. Especially when we do not just use the simple strip but also allow adding extra triangles at each side, as in Figure 19. This strip is used to produce spiralling cylindrical shape of Figure 20.

Figure 19: *Strip of triangles.*

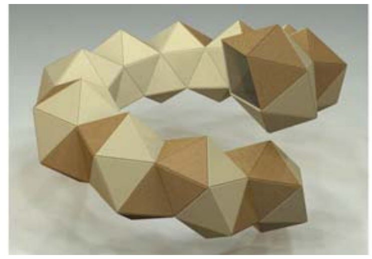


Figure 20: *Spiralling cylinder.*

Many variations are possible. The examples shown in Figure 20 to 22 are all created while starting with a simple strip of triangles on which we add some extra triangles on each side.

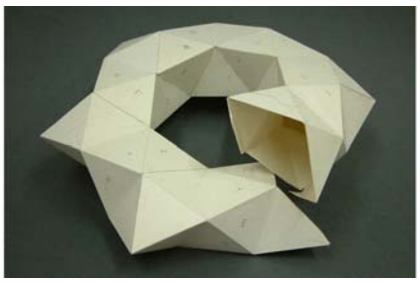


Figure 21, 22: *Spiralling cylindrical tilings.*

3. Twisted Elevation

3.1. Twisting. Until now the tiles we have used can also be applied for flat tilings, or tiles that we knew as faces of the regular solids, the semi-regular solids and their duals. So we can ask ourselves whether it is possible to find other shapes of tiles that only can be used for non flat tilings. To study this we first go back to one of Leonardo da Vinci's drawings of an elevation of a polyhedron, the elevated icosahedron of Figure 23. The step we will now use by making a change in the shape of the tiles is called twisting. The elevated icosahedron can be seen as a set of twenty triangular pyramids. When we rotate each pyramid on it's own axis with the same angle we will get a new object, like the one in Figure 24 and 25. We will call this process twisting. In every stage the object we get is built out of (twenty times three) sixty faces, or tiles. The shape of the tile has changed from an equilateral triangle into a non-convex pentangle. And in every stage the shape of the tiles is the same. The object in Figure 25 can be recognized as a compound of five tetrahedral, but here it just is a set of sixty equal tiles that makes a non-flat tiling.

Figure 23: *Elevated icosahedron.*

Figure 24: *Twisting – first step.*

Figure 25: *Twisting – final step.*

3.2. New shapes. With twisting we have found a method to create new shapes that can be used for non-planar tilings. In general the shape of the resulting tile will be a non-convex polygon. The result of twisting applied on the non-planar tiling, that we created by elevating the planar tilings with squares is shown in Figure 26. And in this case it is clear that the tiles can not be used to make flat tiling. There is no way to fill the plane with these tiles without leaving gaps or without overlaps.
Two more examples can be seen in Figure 27 and 28. The non-flat tiling of Figure 27 is derived from the elevation of the flat tiling of equilateral triangle.

Figure 26: *Twisted elevation.*

Figure 27: *Non-flat tiling.*

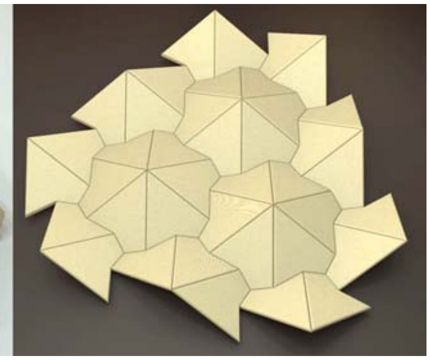


Figure 28: *Non-flat tiling.*

4. Spiral Cylinders

4.1. New shapes for cylindrical tilings. To create new shapes of tiles for cylindrical tilings the method used to generate new tiles described in chapter 3 didn't turn out to be very successful. Therefore a new approach was developed. First a normal spiral curve is drawn. This curve is divided in equal parts. After that a straight line is drawn from the start point of the first part to the start point of the next part. And so on. These straight lines are then extruded downwards to the axis of the spiral, as in Figure 29. This is the basic shape that is needed to construct the shape of the tiles for the cylindrical tiling. The shape of Figure 29 is now turned upside down and added to the original shape (Figure 30). Both shapes do intersect, and from the intersection lines we can derive the final shape of the tile. The completed tiling is shown in Figure 31. The shape of the tile is a non-convex hexagon and can not be used to tile a flat plane. New tilings are created this way.

Figure 29: *Extruded spiral sections.*

Figure 30: *Intersecting 'spirals'.*

Figure 31: *Non-flat cylindrical tiling.*

4.2. Variations. The method described in chapter 4.1 allows many variations: the distance between the point on the spiral, the angle of the extrusion, and the position of the second spiral shape (height as well as rotation angle) may vary. Each set of values will present another shape of tile. In Figures 32 to 34 just a few of the possibilities are shown.

Figure 32: *Cylindrical tiling..*

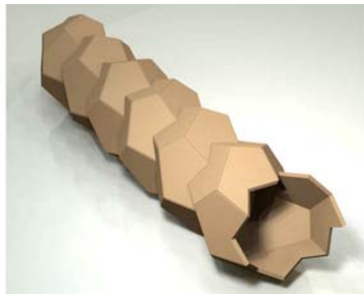


Figure 33: *Cylindrical tiling.*

Figure 34: *Cylindrical tiling.*

4.3. Unroll. Just like the spiralling cylinders in chapter 2, also the objects shown in Figures 31 to 34 can be unrolled into a simple strip of tiles. As an example the object of Figure 34 is unrolled here (Figure 35).

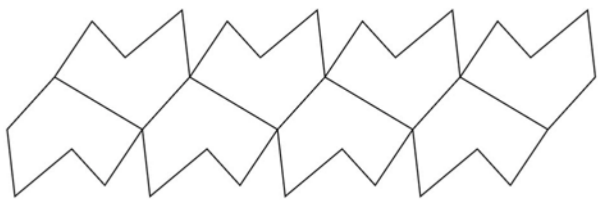


Figure 35: *Plan for cylindrical tiling of Figure 34.*

5. Classification

5.1. Glide Rotation. For the classification of tilings we make use of the symmetry operations that are needed to map one tile of the tiling onto another tile. In flat tilings these operations are called translation, rotation, reflection and glide reflection. When we want to map a tile onto another tile in one of the cylindrical tilings shown in Figure 31 to 34, none of the operations will give the result that we want. Translation, nor rotation, nor reflection, nor glide-reflection will map one tile onto another. What we need here is a combination of two operations, which are translation and rotation. While also glide reflection is a combination of two operations (translation and reflection), the most logical solution seems to be that we introduce a new operation: Glide Rotation, in literature often referred to as “screw rotation”.

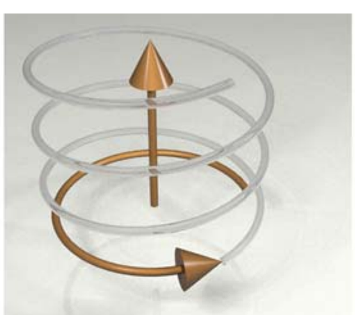


Figure 36: *Glide Rotation.*

Figure 37: *Pentagonal tiles.*

5.2. Polygons. In non-flat tilings all the tiles are polygons. Curved edges is not possible because of the use of flat tiles. An edge between two flat tiles, not laying in a plane is part of the intersecting line of the two planes in which the tiles are laying. And therefore the edge is always a straight line. So another property that we could use for classification might be the number of sides of the polygon. And because convex as well as non-convex polygons can be used, also the position of the non-convex angle can also be used for classification. In Figure 38 some of the main types of tiles for non-flat cylindrical tilings are put together. The notation is according the way it's being described in Heesch's and Kienzle's book Flächenschluss [3] in which they present the types for normal flat tilings.

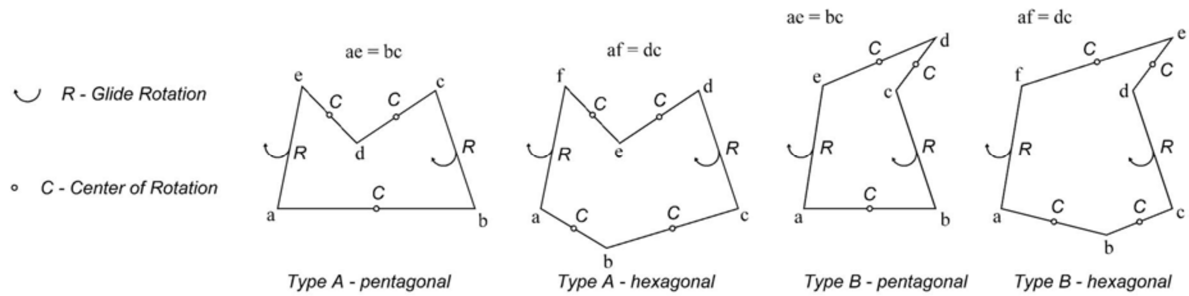


Figure 38: *Classification.*

5.3. Type B. In the pictures 39 to 41 you can see examples of the use of tiles B-pentagonal (Figure 39, 40) and tile B-hexagonal (Figure 41). The concept of non-planar tilings leads to new interesting structures. The method to create cylindrical tilings with flat tiles described in chapter 5.1 is one of the methods I found. To create the tilings shown in Figure 37 and Figures 39 to 41 other methods had to be used.

Figure 39: *Cylindrical tiling - pentagons.*

Figure 40: *Top view.*

Figure 41: *Cylindrical tiling – hexagons.*

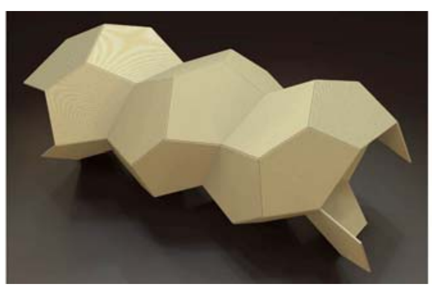


Figure 42: *Convex pentagons - Type C.*

We will use the classification 'Type C' for tilings in which the tiles are convex polygons. Figure 42 shows a cylindrical tiling of Type C with the use of convex pentagons.

5.4. Triangular tiles. In normal flat tilings it is not required that all the corner points of the tiles meet in one point. In ‘Tilings and Patterns’[1], page 475-481 we can see many examples of polygonal isohedral tilings in which the corner point of one tile touches another tile somewhere at the edge in between two corner points. In Figure 43 such a tiling is shown made by M.C. Escher [4]. Also in non-flat tilings this way of connecting the tiles can be used. In the examples (Figure 44 and 45) the constructions are tilings with Glide Rotation.

Figure 43: *M.C. Escher.*

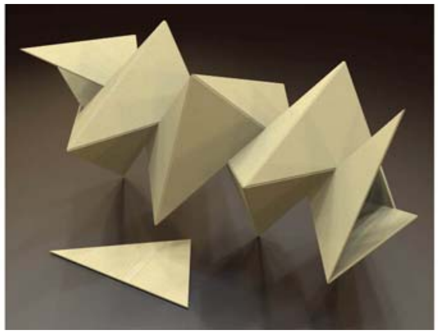


Figure 44: *Triangular tiles (1).*

Figure 45: *Triangular tiles (2).*

5.5. Quadrangular tiles. For convex quadrangular tiles we now have two different ways of using them in non-planar tilings with Glide Rotation as can be seen in Figure 46. In Figure 46 also the triangular tile used in Figures 44 and 45 is shown.

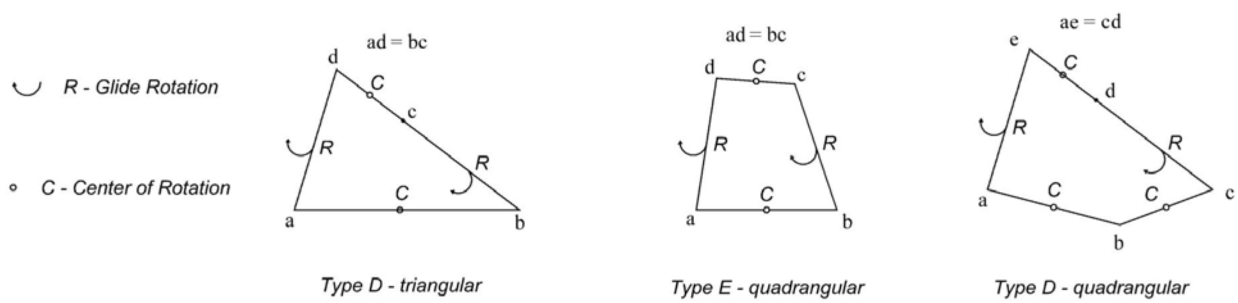


Figure 46: *Classification – convex triangular and quadrangular tiles.*

The use of quadrangular tiles leads to nice constructions. Many variations are possible as can be seen in the examples of Figures 47 to 49.

Figure 47: *Type E - quadrangular.*

Figure 48: *Type E – quadrangular.*

Figure 49: *Type D – quadrangular.*

5.6. Special cases. Finally I want to show some special cases of the spiral cylinders with quadrangular tiles. For the tile of Type E we can change the shape of the quadrangle in such a way that it becomes a rectangle. And now the top view of the construction shows an equilateral triangle as the shape of the hole.

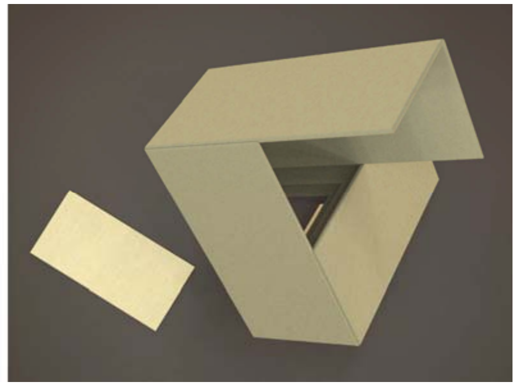
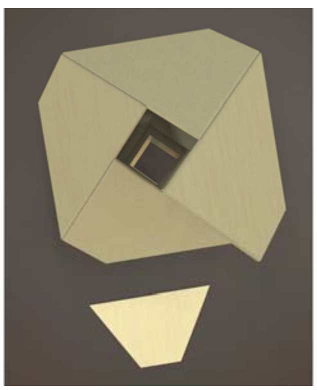


Figure 50: *Quadrangular tiles.* **Figure 51, 52**: *Rectangular tiles.*

And when we use a trapezoid with angles 60-60-120-120 degrees we will get a square hole.



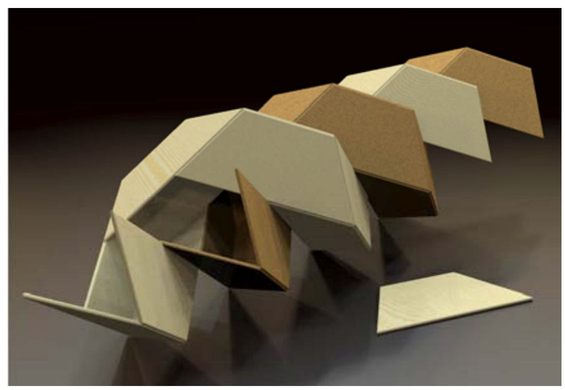


Figure 53, 54: *Trapezoids.* **Figure 55**: *Two entwined spirals.*

A special case of such a trapezoid is the tile used in the construction of Figure 55. The tile is the exact half of a regular hexagon, and it leads to a non-flat tiling that also can be seen as a combination of two entwined spirals.

6. Conclusion

6.1. Non-flat Tilings. I think the field of non-flat-tilings with flat tiles is interesting and needs further exploration. Especially the use of Glide Rotation leads to interesting new constructions which can be used as designs for sculptures. In other words, it is a topic which can bring art and mathematics together in an elegant way.

References

[1] Grünbaum and Sheppard, *Tilings and Patterns*, W.H. Freeman and Company, New York, 1987.
[2] Luca Pacioli, *La Divina Proportione*, Edicione Akal, Madrid, 1991 (first published in 1509).
[3] Heesch und Kienzle, *Flächenschluss*, Spinger-Verlag, Berlin, 1963.
[4] Doris Schattschneider, M.C. *Escher: Visions of Symmetry*, Harry N. Abrams, 2004 (2nd ed.).