Counterchange Patterns and Polyhedra

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Abstract

Recent research has examined the difficulties encountered when attempting to apply two-dimensional repeating designs to wrap around the surface of polyhedra. The study was concerned with symmetry in pattern but did not consider symmetries that involve a color change. A pattern is said to have color symmetry when it exhibits, as a minimum, one symmetry that is color-changing. Counterchange designs are produced when the color-changing symmetries of a pattern involve only two colors. This paper discusses the problems involved in the application of counterchange patterns to polyhedra, focusing particular attention on the icosahedron.

Introduction

Recent investigations have explored the problem of patterning certain polyhedra in a systematic and complete way [1, 2, 3]. The initial inquiry identified which of the seventeen two-dimensional repeating pattern classes are capable of regular repetition around the Platonic solids, applying only the restriction that the pattern's unit cell must repeat across the surface of the polyhedron in exactly the same manner as it does in the plane [1, 2]. Focusing on the application of an area of the pattern's unit cell to act as a tile on the faces of the polyhedra, it was shown that only certain pattern classes are suited to the precise patterning of each Platonic solid. The investigation highlighted the importance of the pattern's inherent lattice structure and the symmetry operations of both the pattern and the polyhedron that were most significant to the process. The application of pattern to the dodecahedron required a different approach due to the impossibility of a regular five-sided figure tiling the plane [2, 3]. Through the exploitation of the concept of duality, a method was presented by which pattern can be applied to the dodecahedron through projection from a related patterned polyhedron. This projection method can also be used to pattern other solids dependant on their inter-relationships [3].

The work outlined above was concerned only with pattern symmetries that do not involve color change. This paper will explore the problems associated with the application of color counterchange patterns to polyhedra. Specific attention will be focused on the difficulties encountered when pattern is applied to the icosahedron in such a way that pattern's unit cell (and constituent coloring) repeats in exactly the same manner around the solid as in the plane.

Counterchange Patterns

A *color symmetry* of a tiling or pattern, with two or more colors, is a symmetry of the (uncolored) pattern that induces a permutation of the colors. Symmetries that consistently interchange or preserve color, are said to be *consistent with color* [4]. A color-changing symmetry that systematically interchanges only two colors is known as *counterchange symmetry*. This characteristic is also known as *antisymmetry* [5]. An infinite checkerboard is a typical example of a pattern that exhibits counterchange symmetry. The elements of the pattern of the one color and the elements of the pattern of the other color are exactly the same. The pattern is therefore made up of two identical components and at least one symmetry of the pattern interchanges the two parts. These patterns are often referred to as *counterchange patterns* in art and design literature [6, 7, 8].

The classification of patterns and tilings, taking into account their color symmetry, is comparatively recent in relation to the classification of designs by geometric symmetry. H.J. Woods [7] was the first to produce complete enumerations the 17 one-dimensional counterchange patterns and the 46 two-dimensional counterchange patterns. This visionary work anticipated the later theoretical developments of crystallographers and mathematicians worldwide [9]. An array of literature has since been published on the subject of color symmetry. Schwarzenberger [10] identified over 100 research papers or other publications dealing with color symmetry from a mathematical perspective. The works of Washburn and Crowe [4] and Schattschneider [11] are more readily accessible to a non-mathematical audience.

Although there is no universally accepted notation, the *type/sub-type* notation developed by Coxeter [12] is currently the most commonly used amongst mathematicians and is the notation used throughout this paper. The pattern *type* is determined by considering all the symmetries of the pattern, both those that preserve color and those that interchange color. The pattern *subtype* is determined by the symmetry formed by one of the colors alone. Another modified form of the *pxyz* notation, proposed by Belov and Tarkhova [13], is commonly used by artists and designers. In this notation a prime (') is attached to the symbol if the corresponding operation is associated with a color change, although there are several exceptions to this rule.

Counterchange Patterns on Polyhedra

Mathematical coloring problems relating to polyhedra are usually concerned with the minimum coloring requirements, the number of distinctly different coloring solutions and coloring to preserve certain color combinations. The problem proposed in this paper refers to coloring a polyhedron through the application of a two-dimensional counterchange pattern to repeat across the faces and edges of the solid, rather than the application of color directly to the polyhedron's faces. A pattern must be wrapped around a polyhedron in such a way that the unit cell of the underlying (uncolored) pattern repeats across the polyhedron's faces in exactly the same manner as it does in the plane. In addition to this, the counterchange symmetry, indicated by the pattern's subtype, must also correspond and repeat in an identical way as in the two-dimensional pattern. This allows for color-changing and color-preserving translations of the unit cell in accordance with the symmetries of the plane pattern. The difficulties encountered primarily relate to the removal of certain areas within the pattern when it is folded into different planes to wrap around the solid. In some cases, the removal of prescribed areas within the pattern cause adjacent motifs and/or color not to register at the polyhedron's edges.

Previous studies [1, 2] have shown that it is possible to pattern the icosahedron with two-dimensional pattern classes p6 and p6m. These patterns are constructed on a hexagonal lattice in which the unit cell comprises of two equilateral triangles. When the patterns are wrapped around the icosahedron, an area equivalent to half the unit cell is applied to each face, ensuring repetition across each of the solid's edges and vertices in exactly the same manner as in the planar pattern. The difficulties encountered with the application of counterchange patterns, based on the two-dimensional pattern classes p6 and p6m, to wrap around the icosahedron are discussed and illustrated below.

Symmetries of the Patterned Icosahedron

When considering the systematic application of two colors to pattern classes p6 and p6m, there is only one possible counterchange coloring of class p6 (p6/p3), while class p6mm offers three possible counterchange colorings (p6m/p3m1, p6m/p31m and p6m/p6). Of these counterchange pattern classes only class p6m/p6 is capable of patterning the icosahedron in such a way that the pattern, and its constituent color symmetries, repeat in the same manner as in the plane (see Figures 2, 3 and 4). The difficulties encountered when applying counterchange patterns p6/p3, p6m/p3m1 and p6m/p31m are indicated in Figure 1. The edges of the patterned net that will not correspond, and do not maintain the color symmetry of the plane pattern, are indicated with an asterisk (*).

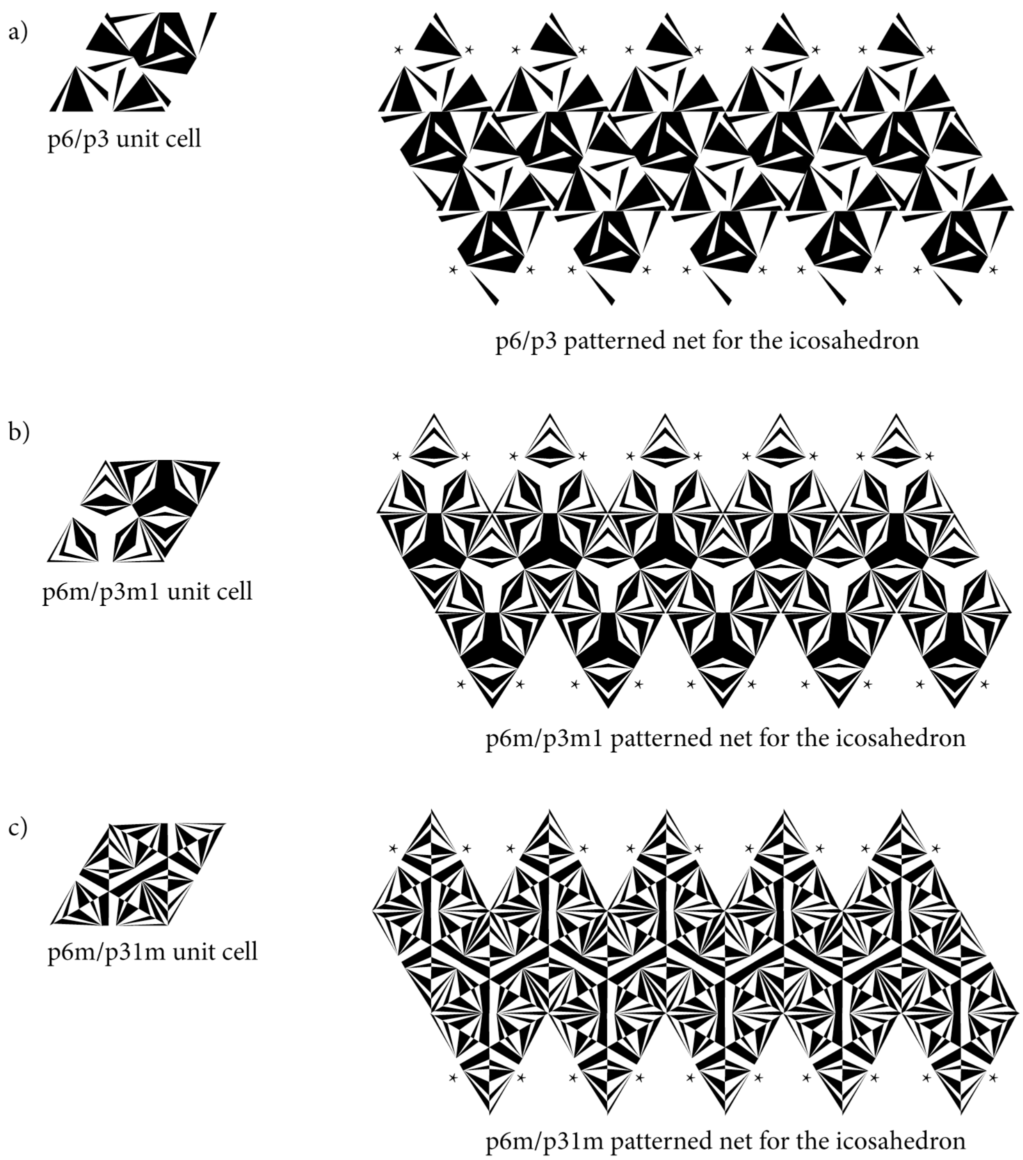
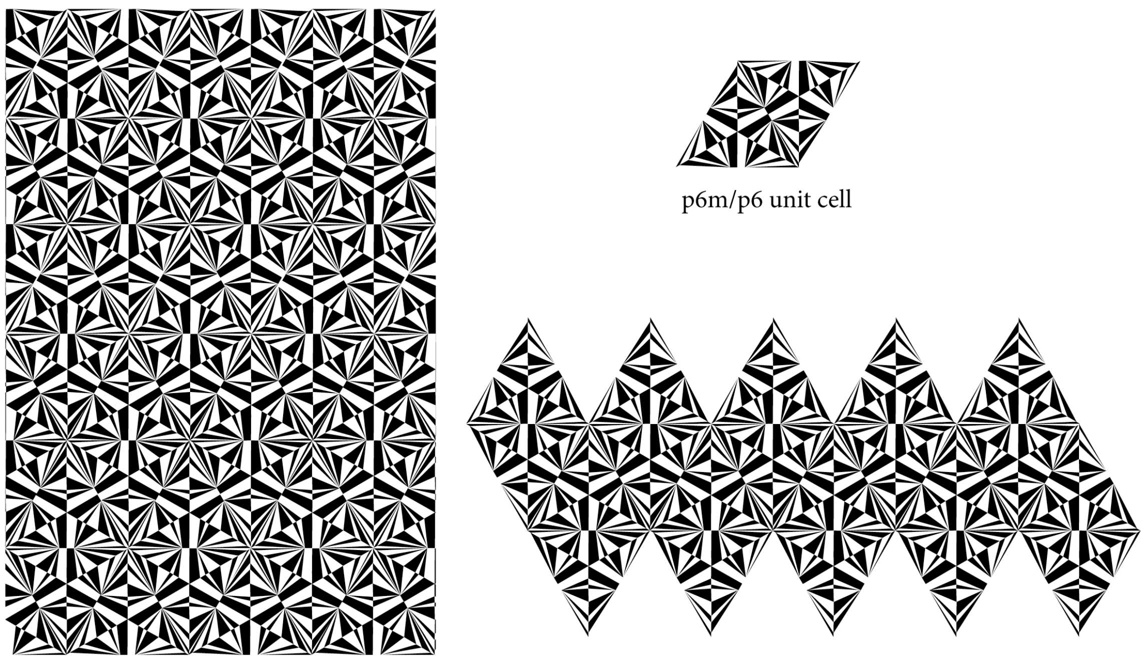


Figure 1 *Nets for the icosahedron patterned with counterchange classes a) p6/p3, b) p6m/p3m1 and c) p6m/p31m, indicating the edges of the patterned polyhedron where the two-dimensional color symmetries are not maintained*

Whilst the underlying symmetry of two-dimensional pattern classes p6 and p6m allow these patterns to be successfully wrapped around the icosahedron, the introduction of color restricts the patterning possibilities. Axes of two-fold and six-fold rotation in pattern p6/p3, shown in Figure 1(a), interchange color in the two-dimensional pattern and the translation of the unit cell preserves color. The removal of 10 prescribed areas of pattern, each equivalent to half the unit cell, is necessary to allow the plane pattern to repeat around the five-fold rotation axes present at the vertices of the icosahedron. As a result, the color symmetries of pattern p6/p3 not corresponding at edges where areas of pattern have been removed. The application of counterchange pattern p6m/p3m1 to the icosahedron presents a similar difficulty, as shown in Figure 1(b). A comparable problem also occurs with the color-preserving reflection planes of two-dimensional pattern p6m/p31m, shown in Figure 1(c). The reflection planes coincide with the edges of the icosahedron, causing the color symmetry of the pattern to conflict at the 10 edges where areas of the pattern have been removed.

The successful patterning of the icosahedron with counterchange pattern class p6m/p6 is illustrated in Figures 2 – 4 and the symmetries of the patterned solid described below.



counterchange pattern class p6m/p6   p6m/p6 patterned net for the icosahedron

Figure 2 *Counterchange pattern class p6m/p6 applied to a net for the patterned icosahedron*

Full icosahedral symmetry (or *achiral icosahedral symmetry*), I_h, exhibits 120 symmetries including transformations that combine reflection and rotation. The archiral icosahedral symmetries are: identity; 12 axes of five-fold rotation; 12 axes of two-fold rotation about a five-fold axis; 20 axes of three-fold rotation; 15 axes of two-fold rotation; 12 axes of rotoreflection by 108 degrees; 12 axes of rotoreflection by 36 degrees; 20 axes of rotoreflection by 60 degrees and 15 planes of reflection.

As the reflection axes of the plane pattern coincide with the 15 reflection planes of the icosahedron, all three-dimensional reflection symmetries are preserved in the patterned solid. Color is interchanged across these reflection planes in the same manner as in the two-dimensional counterchange pattern. Icosahedral axes of two- and three-fold rotation correspond with centers of two- and three-fold rotation in the plane pattern. Centers of six-fold rotation in the plane pattern correspond with axes of five-fold rotation at the patterned solid's vertices, where areas equivalent to half the unit cell are removed as the pattern is wrapped around the polyhedron. All five-fold, three-fold and two-fold axes of rotation on the patterned icosahedron preserve color, whereas all axes of rotoreflection interchange color. It should be noted that while, in this instance, the three-dimensional symmetries of the icosahedron are also symmetries of its patterned faces, it has been shown that in many cases [1, 2] there are two-dimensional symmetries of the plane pattern that are not three-dimensional symmetries of the patterned polyhedron. It is the manner in which a polyhedron is wrapped with a pattern that determines which planar symmetries coincide with the symmetries of the solid.

Figures 3 and 4 show illustrations of the icosahedron patterned with counterchange class p6m/p6. The model shown in Figure 3 is based on the design previously illustrated in Figure 2, where the area of the pattern that is applied to each face is equivalent to half the unit cell. In Figure 4 an area of patter equivalent to two unit cells is applied to pattern each face. In both instances, all faces are identical.

Figure 3 *The icosahedron patterned with counterchange pattern class p6m/p6 (created from laser-etched and painted acrylic)*

Figure 4 *The icosahedron patterned with counterchange pattern class p6m/p6 (created from laser-etched wood composite)*

In Conclusion

This paper builds on knowledge from previous research that presented a method by which appropriate pattern types can be identified and applied to the Platonic solids [1, 2, 3]. The systematic addition of color to a pattern, in association with its symmetries, introduces additional restrictions to the pattern classes that are capable of wrapping the icosahedron in such a way that the pattern repeats in an identical manner as in the plane. The introduction of color symmetry increases the importance of the operations of rotation and/or reflection that coincide at the edges of a patterned solid. It is the color-change associated with these symmetry operations that determines whether color interchanges, or is preserved, correctly between the faces and in exactly the same manner as in the plane pattern.

Further investigation into the application of counterchange patterns to the remaining four Platonic solids should be undertaken to enable a complete analysis of the symmetries of importance to the patterning process. Further research should be undertaken to identify additional areas within the unit cell that are capable of patterning polyhedra in a similar manner as in the plane. Particular attention should be focused on developing methods to apply counterchange patterns to tile the dodecahedron. This inquiry could also be extended to take account of the Archimedean solids and more complex polyhedra, experimenting with patterning through projection [3] and investigating other patterning methods. Future research could also take into account patterns with higher color symmetries.

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