Symmetry and Transformations in the Musical Plane

Vi Hart
http://vihart.com
E-mail: vi@vihart.com

Abstract

The musical plane is different than the Euclidean plane: it has two different and incomparable dimensions, pitch-space and time, rather than two identical dimensions. Symmetry and transformations in music have been studied both in musical and geometric terms, but not when taking this difference into account. In this paper we show exactly which isometric transformations apply to musical space and how they can be arranged into repeating patterns (frieze patterns and variations of the wallpaper groups). Frieze patterns are created intuitively by composers, sometimes with timbral color patterns or in sequence, and many examples are shown. Thinking about symmetry in the musical plane is useful not just for analysis, but as inspiration for composers.

Introduction

Composers have been using symmetry for hundreds of years. Sometimes this is done on a small scale, such as when the notes in one measure of a Bach fugue are found repeated upside-down within the next measure, or when an arpeggio is played going up and then back down. On a large scale, one can find examples of constructions such as crab canons—where the same notes are played forwards and backwards at the same time (most famously in Bach's *A Musical Offering*)—or table canons—where the same sheet music is played by two instrumentalists looking down at it from either side of a table, so that their notes are upside-down and backwards (examples can be found by Bach, Mozart, and others). These simple geometric transformations, containing just one mirror or point of rotation, are well studied and understood by both mathematicians and music theorists.

When looking at a piece, music theorists are interested in following the development of themes and motives. A motive (here used to mean a set of notes) can be found repeated (translated horizontally), transposed (translated vertically), inverted (horizontal mirror), in retrograde (vertical mirror), or in retrograde inversion (180° rotation). We show that these transformations and three other combinations are all of the possible isometries in the musical plane. We precisely characterize which patterns are symmetric under one or more musical transformations: the seven frieze patterns, each mapped horizontally or vertically, and the 14 musical wallpaper groups (not all of the Euclidean wallpaper groups appear, and some appear twice—once horizontally and once vertically). We also account for the symmetry found when passing a motive back and forth between instruments through color patterns, and we analyze sequences by thinking of them as a frieze pattern on a diagonal and relating them to color patterns.

Knowledge of these patterns may be useful for music analysis, but more importantly it is helpful for composers to be aware of the symmetries that we are already using intuitively, so that we can have conscious control over our work.

Related Work. Connections between mathematical and musical symmetry have been studied before [2, 3]. They focus on the basic musical transformations and on frieze patterns, but they do not look systematically at all transformations and combinations of transformations that are possible in the musical plane—in particular, they do not consider color patterns, sequences, or wallpaper groups.

Transformations in Unequal Dimensions

Transformations. A set of notes can be thought of as a set of points in 2-dimensional space. The musical plane, however, has a very important difference from the usual geometric plane: the two dimensions are not the same. Time is seen horizontally while pitch space is mapped vertically, and the fact remains that time is different from space, not just in physics (as in General Relativity's spacetime), but most strongly in our perception. The musical plane therefore has less symmetry than the Euclidean plane—for example, rather than having symmetry at any mirror line, the musical plane only has horizontal and vertical axes of symmetry. However, this very limitation also splits most Euclidean transformations into two unique musical transformations, one horizontal and one vertical.

Isometries. In the Euclidean plane, there are three types of transformations that preserve the distances between all points or notes (isometries): translations, reflections, and rotations. In the musical plane, all transformations must treat each of the two dimensions separately, leaving these possible isometries: horizontal translation (repetition), vertical translation (transposition), horizontal reflection (retrograde), vertical reflection (inversion), and combinations thereof. A rotation of 180° can be done, though in music terms this is thought of as a combination of the two mirrors and called a retrograde inversion. Likewise, the horizontal and vertical translations can be combined to repeat a motive in a different key (also called transposition). The last combination is the glide reflection (a translation plus reflection), which also comes in two types: if the glide reflection axis is horizontal, an inversion follows the motive, while if it is vertical the motive is followed by its retrograde in transposition. Any two isometric instances of a motive are related by exactly one of these eight geometric transformations.

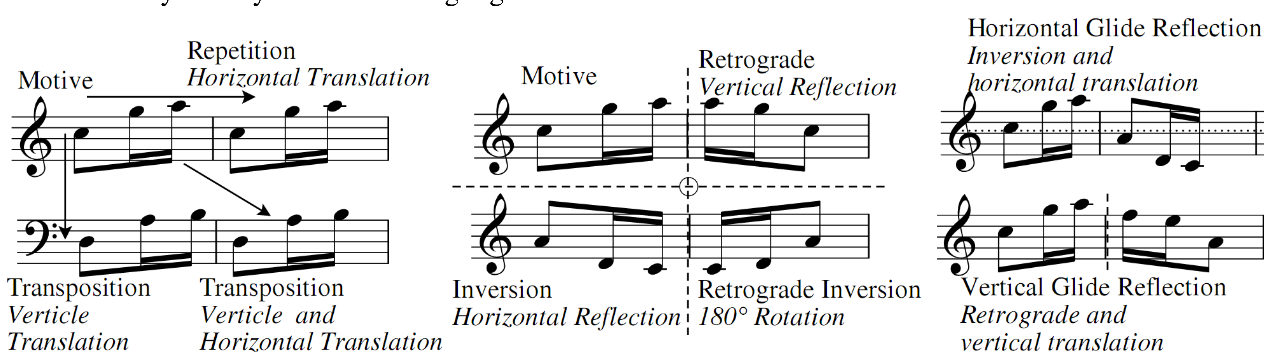


Figure 1: *All Eight Isometric Transformations in Musical Space*

Choosing a scale for pitch. Pitch, aside from being wholly different from time, can be measured in various ways. Most music is written tonally within some scale, and so it makes sense to use one step of that scale as one unit of pitch when applying transformations. In music, this is called a *tonal* transformation. This is the metric used in the examples in this paper because it is most common and easiest to see visually in music notation. On the other hand, a *real* transformation uses each step of the chromatic scale as one pitch unit. There are comparatively fewer musical examples of this second type (though still very many, especially in 12-tone music).

Frieze Patterns

Frieze patterns. Frieze patterns are the seven symmetry groups of patterns that repeat infinitely in just one dimension in the Euclidean plane. These patterns often appear as borders in art and architecture, and are especially well suited to music because they share the property that their two dimensions are not equal (one is infinite and one is not). Like music, frieze patterns can contain only repeating symmetries that use translation, glide reflection, horizontal mirrors, vertical mirrors, and 180° rotation, in various combinations. If you count time as possibly infinite and audible pitch space as limited, then frieze patterns exactly match the symmetric capabilities of a repeating phrase of music. Of course, no music can really repeat endlessly in one dimension, just as no frieze patterns in architecture do either, so a few repetitions will have to do.

Frieze patterns are so natural to music that a composer can unknowingly use all seven groups within a single piece. Below, we illustrate each of the seven frieze patterns with an example from Debussy's *Three Nocturnes* on the right, as well as a simple example using the motive from Figure 1, on the left. The seven frieze patterns have various notations, but we find Conway's footprint analogies to be intuitive and easy to use as names [1]. We also give the orbifold notation and describe the groups using both mathematical and musical terms. The analogous feet are included above the corresponding notes.

Hop ∞∞. The hop has translational symmetry only. In music, it is repetition. Repeated motives, bass lines, choruses, chord progressions, and melodies are so common that hops are difficult to avoid in music.

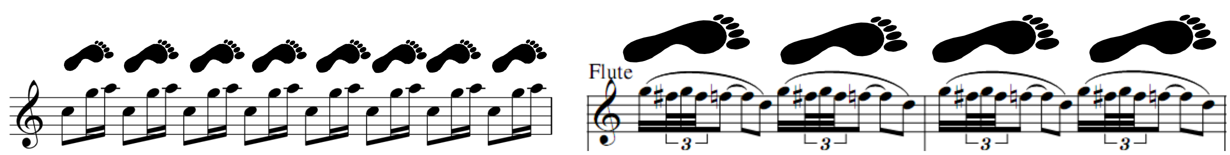


Figure 2: *Hops.*

Step ∞**x.** The step is a glide reflection—a pattern of a motive and its inversion. Bach was especially fond of glide reflections and they can be found in much of his work

Figure 3: *Steps.*

Jump ∞*. A hop with a horizontal mirror makes a jump. A motive and its inversion are played simultaneously, and repeated. Jumps are a difficult frieze pattern to find in music, perhaps because horizontal mirrors do not sound very subtle. Some Bach pieces, for example, contain a horizontal mirror at the end of a piece or section which had many glide reflections, bringing closure to the motive and its inversion by calling attention to that they are really the same under transformation.

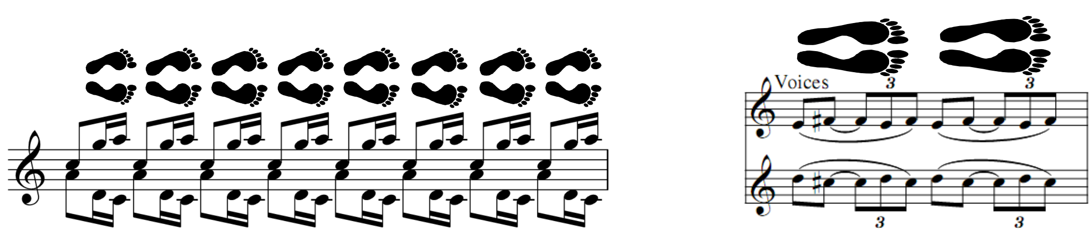


Figure 4: *Jumps.*

Sidle *∞∞**.** A sidle contains two vertical mirrors, alternating a motive with its retrograde. They are commonly found in arpeggios, especially piano accompaniment.

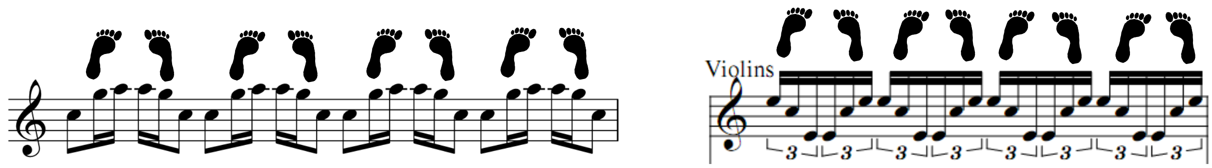


Figure 5: *Sidles.*

Dizzy hop 22∞. The dizzy hop, or spin hop, has two points of 180° rotation. In music, this alternates a motive with its retrograde inversion. There are many frenetically dizzying dizzy hops in Stravinsky's music.

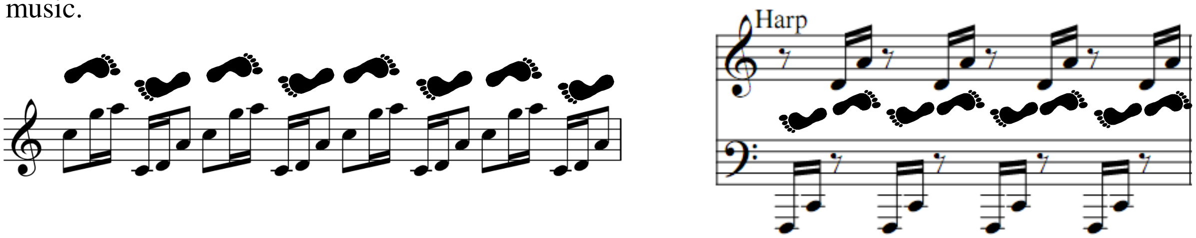


Figure 6: *Dizzy hops.*

Dizzy sidle 2*∞. A sidle with a twist yields the dizzy sidle—one vertical mirror and one point of 180° rotation. It contains all four transformations of a motive in order: a motive, its retrograde, the inversion, and the retrograde inversion, repeated indefinitely. Traditional Alberti Bass has this symmetry (in which the mirror lines go through the top and bottom notes).

Figure 7: *Dizzy sidles.*

Dizzy jump *22∞. The dizzy jump has all possible symmetries: one vertical and one horizontal mirror, which create a point of rotation where they meet. A motive and its inversion are played simultaneously, and then the retrograde and retrograde inversion played simultaneously as well. Like jumps, this is a less common frieze pattern in music—except for music where the horizontal mirror sound is used purposely, such as Philip Glass's music in which dizzy jumps are quite commonplace.

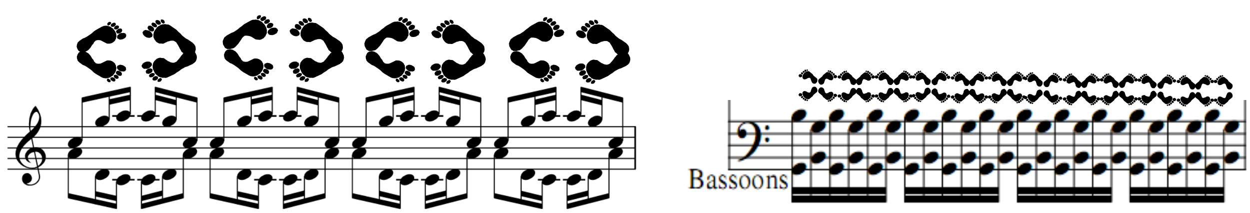


Figure 8: *Dizzy jumps.*

The other seven musical frieze patterns. Though it is intuitive to map frieze patterns to the musical plane in the horizontal (time) dimension, it is still geometrically sound to map a piece of a frieze pattern vertically (pitch). The aesthetic problem with this mapping is that an infinitely long pattern must be suggested within a very short amount of time through the use of a large range of pitches. Time can feel infinite from the perspective of one moment, but pitch-space is severely limited by the amount of information the ear can perceive at once. While examples can be found in music where many instruments in different octaves play the same thing, this is usually done for timbre or volume. We hear all the octaves together as one melody rather than as a repeating pattern through pitch-space, making these vertical frieze patterns not as useful from an analysis or composition viewpoint. Still, it is worth noting that there are actually 14 types of frieze patterns in the musical plane.

Color

In mathematics, the word "color" refers to a labeling. In music, "color" refers to the timbre, or sound quality, of an instrument or set of instruments. Just as a color labeling differentiates sets of points in mathematics, sound color is what lets us hear the difference between sets of notes played by one instrument rather than another. Frieze patterns, both musical and mathematical, have color patterns associated with them [1]. Most frieze patterns can be colored in more than one way. For example, a jump could alternate each set of footprints between two colors, or it could have all the left feet one color and all the right feet another. There are 17 different 2-color frieze patterns [1], many of which—if not all—can be found in existing music, including the previous example of Debussy's *Nocturnes* as shown in Figure 9.

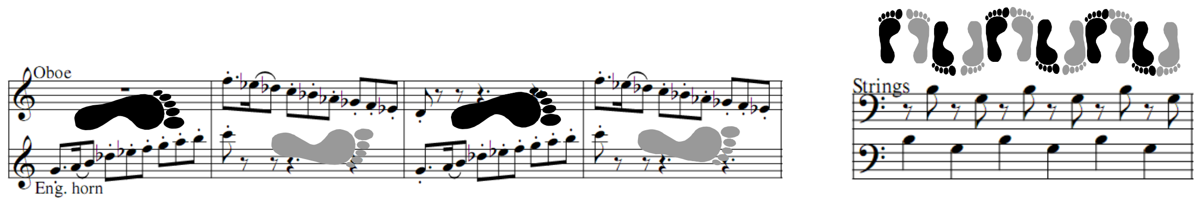


Figure 9: *Debussy Nocturne I has a 2-color step and a 2-color dizzy sidle.*

Sequences

The frieze patterns above are missing one of the most common musical transformations: transpositions involving both vertical and horizontal translation. All frieze patterns contain horizontal translations by definition, and all of the other musical isometries are accounted for as well. But what if each repetition of a motive were translated not just horizontally, but also vertically? In music this is called a sequence, and they run rampant in classical styles. Just as an arithmetic sequence adds a fixed number to each successive number, a musical sequence adds a fixed interval to each successive repetition of a motive.

A sequence can be thought of as a slanted frieze pattern. By normalizing the sequence (vertically translating each repetition to have the same vertical extent), one can then analyze the frieze pattern. The difficulty lies in picking the repeating unit. In the second example in the figure below, one can normalize each triplet in order to get a sidle. However, if instead of taking the first three notes, one took three notes starting with the second, then the sequence would normalize to a dizzy hop. It is usually easy to find the musical cues necessary to decide what the motive is.

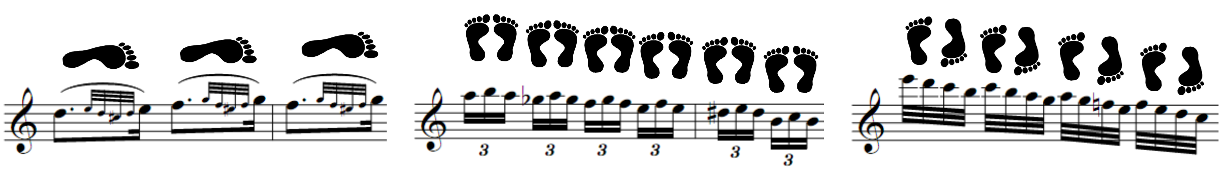


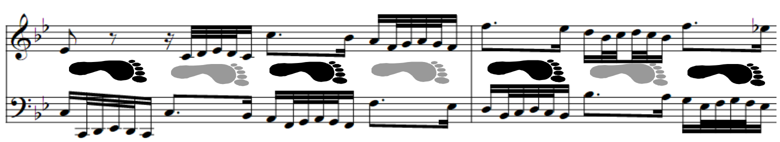
Figure 10: *Three sequences from Mozart's Piano Sonata in C: a walk, sidle, and dizzy hop.*

The repeating unit of a frieze patterns consists of one, two, or four transformed instances of the motive (feet). When creating a sequence from a frieze pattern, the vertical shift can be applied every one, two, or four instances of the motive (up to the length of the repeating unit), and each choice will create a unique symmetry group. (One could apply the shift to a larger set of instances, but the result would fall into one of the other groups.) There are eleven different kinds of frieze pattern sequences, a subset of 2-color patterns where each time the color changes, a transposition occurs (the missing six of the seventeen color patterns are the ones where two colors occur at the same point in time—once again the difference between time and space creates a difference between the geometry of music and of the usual plane—and the ones where the color change is at a point where there is a built-in vertical translation through glide reflection or rotation, making it impossible to tell whether there is an extra translation or not). The figure below shows the two different ways of sequencing a sidle.

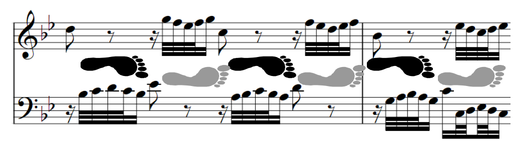
Figure 11: *The two ways of sequencing a sidle, related to the two possible 2-colorings.*

Color and Sequences Together: A Bach Example

Bach's Invention 14 is a great example of color and sequence: it contains hops and steps of the same motive in all possible 2-color patterns between the right and left hand, in sequence. The piece also contains both hops and steps in just one hand and in both hands at once.



2-color hop



2-color step one way

2-color step the second way

Figure 12: *2-color hop and step sequences in Bach's Invention 14.*

Wallpaper Patterns

The wallpaper groups are the symmetry groups of patterns that repeat in two dimensions. There are 17 of them, but not all can be naturally mapped to the musical plane because many contain rotations other than 180° or have mirror and glide-reflection lines which meet at angles other than 90°. The following eight patterns (in orbifold notation [1]) do work: o, **, xx, *x, *2222, 2222, 22*, and 2*22, because they can be mapped onto the musical plane with all mirror lines and glide-reflection axes going either horizontally or vertically, and do not contain rotations other than 180°. Interestingly, just as with the Euclidean isometries, this limitation also splits some wallpaper groups into two unique musical wallpaper groups, because they can be mapped onto the musical plane in two different ways at 90° from each other. *2222 and 2*22 fit just one way each. 22*, xx, *x, and ** find two distinct ways to map themselves to the staff. Patterns o and 2222 can both fit at any angle, though they fit into two classes: the way that makes repetitions play in parallel, and the infinite possible offsets.

To give these pairs distinct names, we add a "V" for vertical mirror, glide-reflection axis, or parallel repetitions, and add "H" for horizontal mirror, glide-reflection axis, or nonparallel repetitions. Figure 13 illustrates swatches of the 14 groups, showing just one vertical repetition.

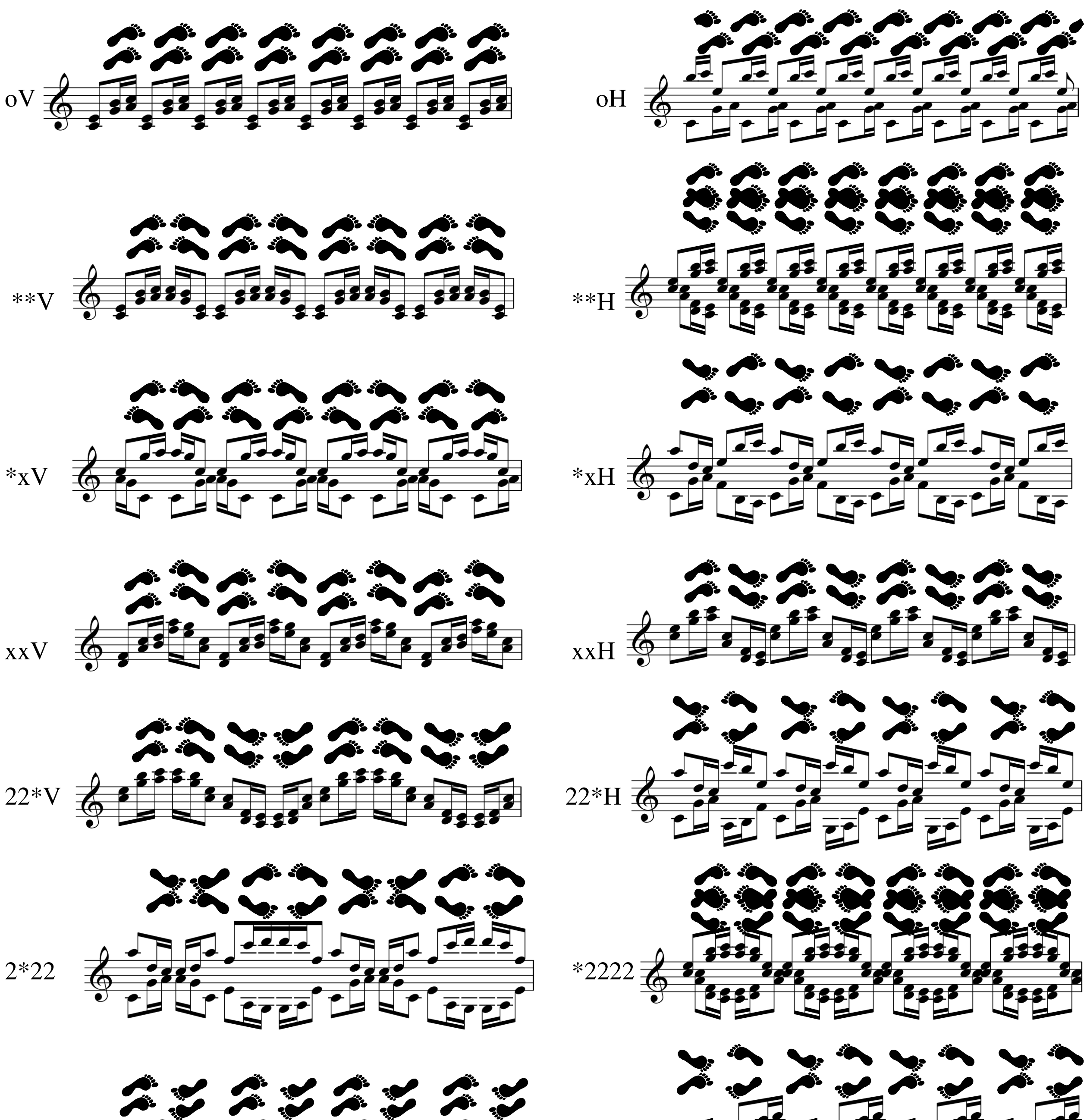


Figure 13: *The 14 musical wallpaper groups*

There is the same number of musical wallpaper groups as musical frieze patterns, but the wallpaper groups cannot all be constructed by simply repeating frieze patterns in the second dimension. However, if one repeats each of the 14 musical frieze patterns with either a translation in the second dimension, a translation in the second dimension followed by reflection, or a translation in both dimensions, the set of possibilities (removing duplicates) is the 14 musical wallpaper groups. Interestingly, though there are many duplicates, both horizontal and vertical frieze patterns are required under these transformations to get all 14 wallpaper groups.

Use in Composition

The author finds these ideas particularly inspiring for use in composition. For example, it seemed obvious to begin a seven-movement piece with one movement dedicated to each of the seven horizontal frieze patterns. This piece, as of yet untitled, is a set of variations where one motive is transformed as it travels through the different symmetries of the movements. The mathematics is used to inspire and inform the pieces rather than to generate them, and so the frieze patterns are used in the melody and accompaniment when it is musical to do so but not conformed to when the author finds it beautiful to do otherwise.

The figure below contains excerpts from the first four movements to show how the motive evolves as it finds itself in different frieze patterns.

Figure 14: *Four frieze variations.*

For recordings and more, see http://vihart.com/papers/symmetry/

References

[1] John H. Conway, Heidi Burgiel, and Chaim Goodman-Strauss, *The Symmetries of Things*, A K Peters, 2008.
[2] Colleen Duffy, "Symmetrical musical pieces," Manuscript, 2001. http://www.math.rutgers.edu/~duffyc/research/SymmetricalMusical2b.doc
[3] Wilfred Hodges, "The geometry of music," in *Music and Mathematics: From Pythagoras to Fractals*, Oxford University Press, 2003, pp. 91–111.