The Eleven–Pointed Star Polygon Design of the *Topkapı Scroll*

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Abstract

This paper will explore an eleven-pointed star polygon design known as Catalog Number 42 of the *Topkapı Scroll*. The scroll contains 114 Islamic architectural and ornamental design sketches, only one of which produces an eleven-pointed star polygon. The design also consists of "nearly regular" nine-pointed star polygons and irregularly-shaped pentagonal stars. We propose one plausible "point-joining" compass-and-straightedge reconstruction of the repeat unit for this highly unusual design.

Introduction

The *Topkapı Scroll,* thought to date from the 15th or 16th century (and so named because it is now housed at the Topkapı Palace Museum Library in Istanbul), is a 96-foot-long scroll containing 114 Islamic architectural and ornamental design sketches. Believed to be the earliest known document of its kind (and also the best-preserved), most of the sketches in this manual show only a small portion, or *repeat unit,* of an overall pattern, contained, for the most part, within a square or rectangle. Thus, to achieve the entire pattern, these repeat units may be replicated by reflection across the boundaries of the rectangles or by rotation about the vertices of the squares. Since this scroll contains no measurements or accompanying explanations for the creation of these idealized Islamic patterns, it is believed by Harvard professor Gülru Necipoğlu, the recognized authority on the scroll and the author of *The Topkapı Scroll – Geometry and Ornament in Islamic Architecture: Topkapı Palace Museum Library MS H. 1956,* that the drawings most likely "served as an *aide-memoire* for architects and master builders who were already familiar through experience with the coded graphic language used in them" [1, pages 10-12].

How, then, using only the traditional drafting tools of the medieval period – the compass, straightedge, and set square [1, page 138] – were the original patterns (from which the repeat units were generated) conceptualized and created? More specifically, how did the designer of the only eleven-pointed star polygon pattern of the *Topkapı Scroll* (known as Catalog Number 42 or CN42 for short) determine, without mensuration, the proportion and placement of the polygons comprising the design? Some hints for answering this question are provided by the scroll itself. This paper will explore these and propose one plausible "point-joining" compass-and-straightedge reconstruction of the repeat unit for this very rare design.

Topkapı Scroll, Catalog Number 42

The original repeat unit for CN42 was drawn in black ink on cream-colored rag paper [1, page 29], a facsimile of which is reproduced in [1, page 309]. The author's reproduction of CN42, made by tracing the image and producing its copy using the *Geometer's Sketchpad* software program [2]), is shown in

Figure 1a. The repeat unit contains two halves of eleven-pointed star polygons along its vertical edges and two halves of nine-pointed star polygons along its horizontal edges. Between the half-star polygons are eight irregularly-shaped pentagonal stars and two arrow-like polygons. Underlying the black-inked drawing of CN42's repeat unit are faintly embossed "dead" construction lines, scratched on the paper with a pointed tool such as a compass or stylus "so as not to detract from the inked patterns they generate" [1, page 31]. **Figure 1b,** which is a reconstruction of what appears on page 255 of [1], shows the inked pattern (as dashed line segments) superimposed on the uninked "dead" construction lines (in bold). Note that all of the star polygons (the "nearly regular" nine- and eleven-stars as well as the irregularly shaped pentagonal ones) drawn in black ink in Figure 1 are constructed using inscribed and circumscribed circles. The remaining polygonal shapes may be achieved by joining certain intersection points with line segments in such a way that the uninked "dead" lines do form the basis for most of the pattern. Also, notice that the pattern in the repeat unit has 180 degree rotational symmetry about the center of the rectangle. This is useful in the construction of the pattern.

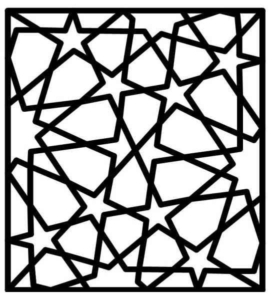

Figure 1a. *Author's reconstruction, of CN42 [1, p 309] produced using the Geometer's Sketchpad software*

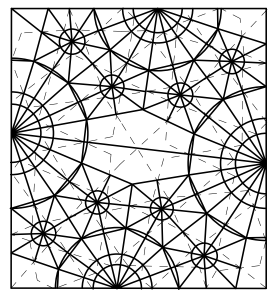

Figure 1b. *Author's reconstruction of the overlay for CN42 [1, p 255] produced using the Geometer's Sketchpad*

The repeat unit of the *Topkapı Scroll's* only eleven-pointed star polygon design is extraordinary for several reasons. First, the vast majority of Islamic star designs have an even number of points, and CN42 has none of these. Second, the repeat unit contains both nine- and eleven-pointed star polygons, presumably created from an underlying grid consisting of nine-gons and eleven-gons. As discussed in more detail in the next section, neither nine-gons nor eleven-gons are *constructible* using the traditional drafting tools of straightedge and compass, so most likely an approximation was utilized to create them. Third, the position of the stars in the repeat unit is remarkable as well. Usually the major stars appear centered at either corner vertices or at the midpoints of the edges in the repeat units of Islamic star patterns in the *Topkapı Scroll*. But for CN42, the centers of the large stars are not located in these standard positions.

Constructing Star Polygon Designs Inscribed within Regular Polygons

One of the most straightforward techniques for creating geometric "heavenly star" patterns found throughout the Islamic world (and for which there is evidence that this method was used [3]) is to initially construct an *n*-gon (a polygon with *n* sides) and then draw in the corresponding regular *n*-pointed star polygon by methodically joining the vertices of the *n*-gon with line segments (diagonals) or by methodically joining midpoints of the *n*-gon's edges. For example, every second vertex of a regular pentagon is connected with line segments to form the regular pentagonal star polygon shown in **Figure 2a** on the following page. By connecting alternate midpoints of edges of the same regular pentagon, a smaller regular pentagonal star polygon is formed, as shown in **Figure 2b**. Similarly, star polygons with *n* points may be created within any *n*-gon, where every second (or third or fourth or k^{th}, where $2 \le k \le n/2$) vertex (or midpoint) is connected.

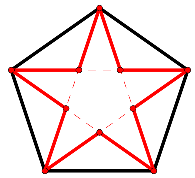

Figure 2a.

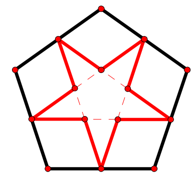

Figure 2b.

The most common geometric regular star polygon designs have *n* points where, *n* is 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, and so on. It should be noted that these regular *n*-gons are *constructible* in the Euclidean sense; that is, they may be constructed using only a compass and straightedge. For $n = 7, 9, 11, 13, 14, 18\ldots$, the regular *n*-gons (and likewise, the corresponding regular *n*-star polygons) may only be constructed approximately using these tools. Of these *non-constructible* star polygons, master builders in the eastern regions of the Islamic world did regularly create star patterns with $n = 7, 9, 14$ and 18.

Construction of a Nearly Regular Eleven-Sided Polygon and its Inscribed Star Polygon

One way to construct a very good approximation of a regular eleven-gon is to follow the method that the author adapted from an origami folding procedure [4]. (For those interested in the mathematics behind this construction, the angle constructed may be calculated to be 0.4 arctan(7), based on a double approximation – first that π is approximately equal to 22/7 and second, that tan (π/22) is approximately 1/7 for small angles. For more information see [4].)

Starting with a square, construct a diagonal and its midpoint. Then draw a line segment from the midpoint to one of the remaining vertices of the square (**Figure 3a**). Bisect this segment with a perpendicular line segment that terminates on the edges of the square (**Figure 3b**). Repeat this procedure twice more so that there are now three perpendicular segments parallel to the diagonal as shown in **Figure 3c.** Now draw a line segment from the midpoint of the square's diagonal to one of the endpoints of the perpendicular line segment constructed last. This segment and one of half of the square's diagonal form an obtuse angle, shown in **Figure 3d**. Bisect this angle as shown in **Figure 3e**. This angle bisector and the line segment that is the other half of the square's diagonal form a second obtuse angle (**Figure 3f**).

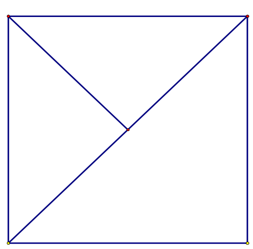

Figure 3a.

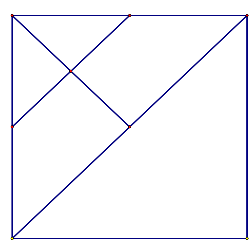

Figure 3b.

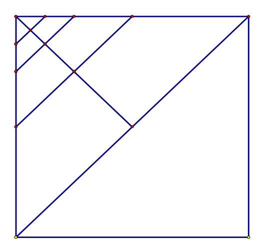

Figure 3c.

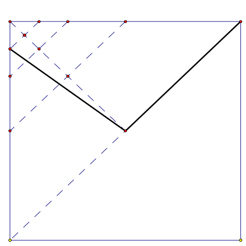

Figure 3d.

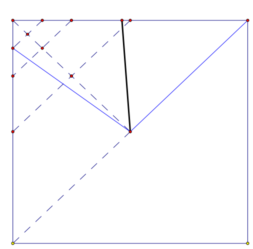

Figure 3e.

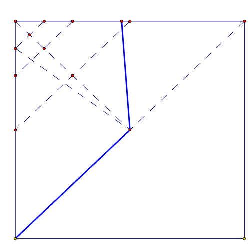

Figure 3f.

Bisect this angle as shown in **Figure 3g,** then bisect the angle that is formed by this second angle bisector and the same segment of the square's diagonal, as shown in **Figure 3h.** The resulting angle (shown in **Figure 3i**) measures approximately 32.733 degrees (as measured by the *Geometer's Sketchpad* software program), which is roughly one-eleventh of a circle's angular measure. (One-eleventh of 360 degrees is exactly 32.727272… degrees, or approximately 32.723 degrees, hence the measure of this constructed angle differs by only one hundredth of a degree from the exact central angle measure.)

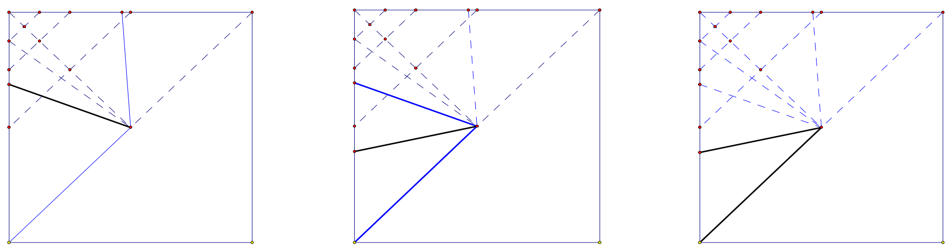

Figure 3g. **Figure 3h.** **Figure 3i.**

Once we have the requisite angle we may replicate it to form an eleven-spoked circle (**Figure 4a**) which leads to the construction of the inscribed eleven-gon (**Figure 4b**). Constructing the midpoints of the eleven-gon's edges (and erasing the circumscribing circle) yields the image in **Figure 4c**.

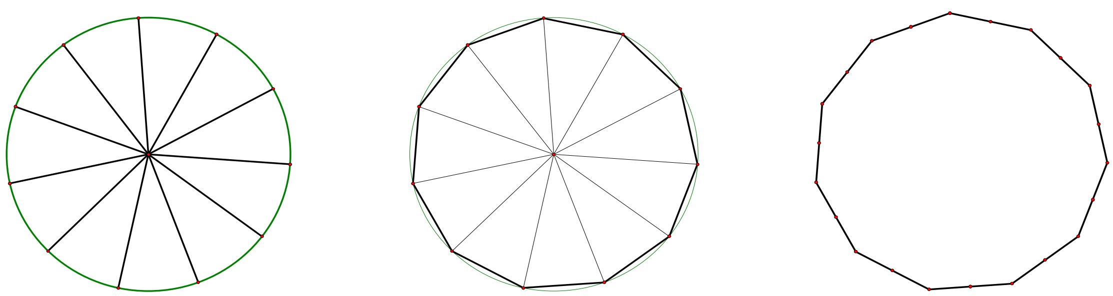

Figure 4a. **Figure 4b.** **Figure 4c.**

The construction of the eleven-pointed star polygon is now straightforward. Connect every 9th point along the perimeter (there are 22 points – 11 vertices and 11 midpoints) as shown in **Figure 4d**. This produces a 22-pointed star, but highlighting every other point, as shown in **Figure 4e** and erasing the remaining dashed line segments yields the eleven-pointed star polygon with the same proportions as those found in the repeat unit of CN42 (**Figure 4f**).

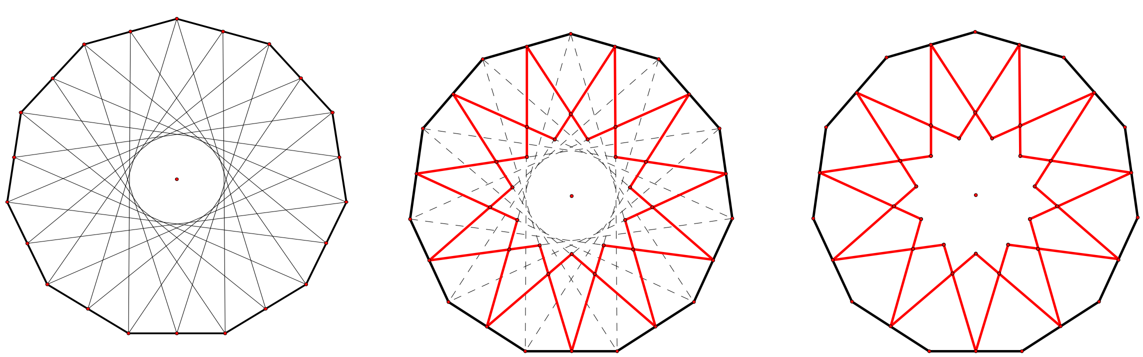

Figure 4d. **Figure 4e.** **Figure 4f.**

A Plausible Construction of the Original Eleven- and Nine-pointed Star Polygon Design

Now that we have constructed a "nearly regular" eleven-pointed star polygon inscribed within an eleven-gon and a circle, we may use this to recreate a very close approximation of the repeat unit shown in CN42 of the *Topkapı Scroll*. Start with a horizontal ray and a vertical ray meeting at a common vertex. These rays will define the top and right edges of the repeat unit rectangle. Construct two congruent circles of any radius, the first centered at the top right corner of the rectangle, and the second tangent to the first circle and centered on the right edge of the rectangle, shown in **Figure 5a**. Construct the eleven-pointed star polygon within the second circle using the method described previously (**Figure 5b**). Now construct three equispaced rays emanating from the center of that polygon by using existing intersection points, as shown in **Figure 5c**.

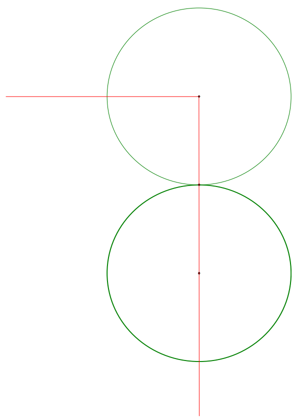

Figure 5a.

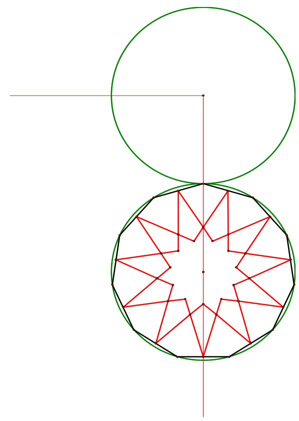

Figure 5b.

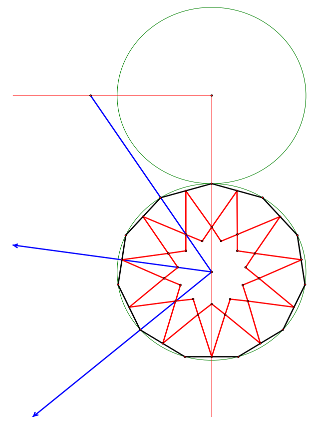

Figure 5c.

The top constructed ray intersects the top of the rectangle. From that intersection point, construct a line perpendicular to the constructed middle ray, as shown in **Figure 5d**. This perpendicular line segment intersects both the middle and lower constructed rays. At the lower intersection point, construct a line parallel to the top edge of the rectangle; this defines the lower edge of the rectangle. The intersection of the perpendicular with the middle ray is the center of the rectangle. With this point as center, construct a circle through the center of the eleven-gon; this circle intersects the middle ray. At that point of intersection, construct a line parallel to the right edge of the rectangle, completing the rectangle (**Figure 5e**). This point of intersection is also the center of the second star eleven-gon, which can be constructed as before (**Figure 5f**). These last two steps exploit the 180 degree rotation symmetry of the repeat unit.

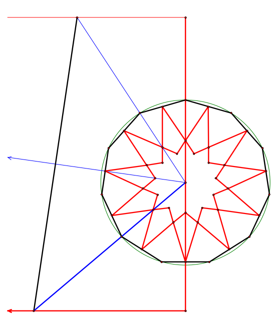

Figure 5d.

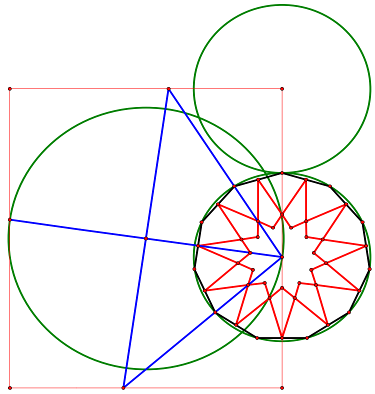

Figure 5e.

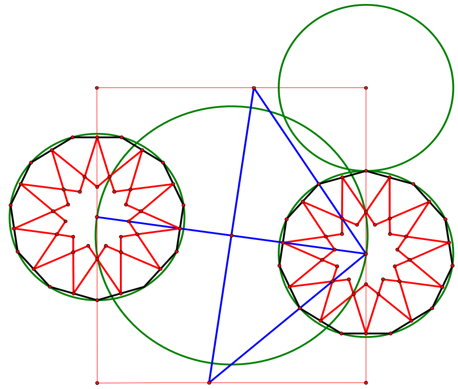

Figure 5f.

Erasing the circles, their inscribed eleven-gons, and the halves of the star eleven-gons outside the repeat unit and constructing two additional line segments from the center of the left eleven-star to the points of intersection of the diagonal with the upper and lower edges of the rectangle yields a repeat unit partially filled with two halves of star eleven-gons and a large rhombus touching each side of the rectangle (**Figure 6a)**. Next bisect the four angles that are formed by the upper edge of the rectangle, the two upper edges and the diagonal of the rhombus (**Figure 6b**). Extend the upper edges of the eleven-stars that are parallel to the edges of the rhombus until they intersect the upper edge of the repeat unit. The point where these lines intersect the long diagonal of the rhombus define the radius of a semicircle centered at the endpoint of that diagonal (**Figure 6c**).

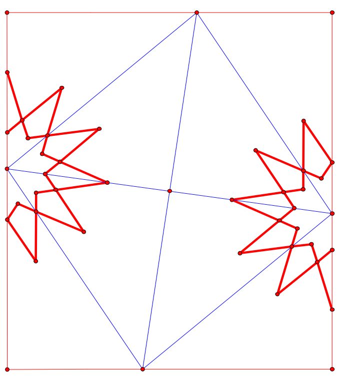

Figure 6a.

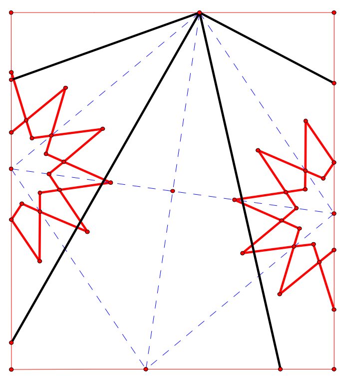

Figure 6b.

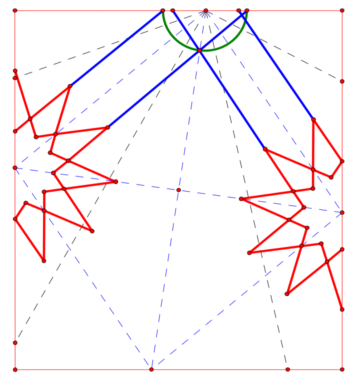

Figure 6c.

The points where the angle bisectors intersect these new line segments define the radii of two additional semicircles, both concentric with the one just created (**Figure 6d**). These three semicircles will be used to construct half of the nine-pointed star polygon. Constructing line segments between certain points on the three semicircles and extending two existing segments that are parallel to the diagonal of the rhombus yields two half nine-star polygons, one within the other (**Figure 6e**). After erasing these semicircles and some segments, construct a line segment from the upper right corner of the rectangle to the midpoint of the long diagonal of the rhombus. The point where this segment intersects the right upper edge of the rhombus defines the radii of two new tangential semicircles, a large one centered at the eleven-star and the smaller one centered at the nine-star. The smaller semicircle also intersects the upper left edge of the rhombus thereby defining the radius of a second larger semicircle centered about the other eleven-star. Construct a line segment through this new point from the midpoint of the long diagonal to the edge of the rectangle. The point where this line intersects the upper segment parallel to the upper left edge of the rhombus defines a second semicircle centered about the nine stars, but slightly smaller (**Figure 6f**).

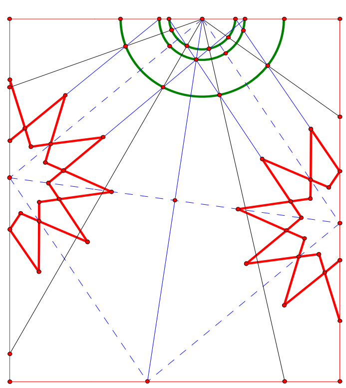

Figure 6d.

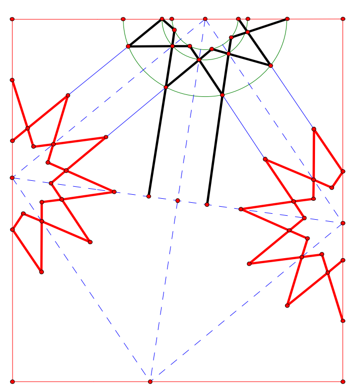

Figure 6e.

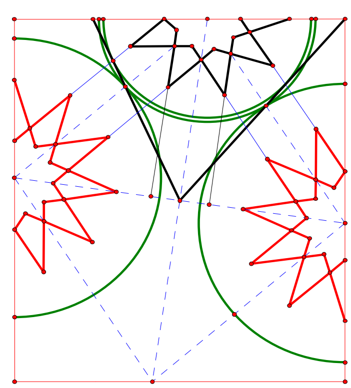

Figure 6f.

To complete the rosettes of the nine-star construct a line from the center of the right eleven-star to the upper left corner of the rectangle. Construct a second line from the center of the left eleven-star through the point of tangency between the two semicircles to the right of the diagonal. Join existing points of intersection between segments and semicircles and extend line segments to produce half of the nine-star rosette (**Figure 7a**). After erasing the now-dispensable semicircles and line segments, construct an additional semicircle, centered about the right eleven-star and defined by the point of intersection between the segment connecting the upper right corner with the midpoint of the rectangle and the innermost segment parallel to the upper right edge of the rhombus. Similarly, a congruent semicircle may also be constructed centered about the left eleven-star. By extending segments and joining existing points, the outlines of the pentagonal stars and the rosettes of the eleven-star rosettes begin to form (**Figure 7b**). Extend two additional segments to construct the upper halves of the two arrow-shaped polygons in the center of the rectangle (**Figure 7c**).

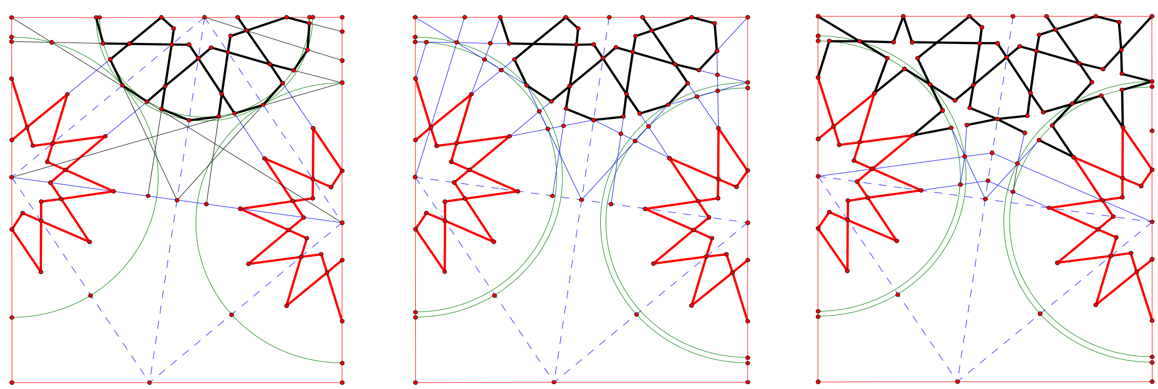

Figure 7a. **Figure 7b.** **Figure 7c.**

Figures 7d and **7e** show the upper half of the repeat unit before and after the now unnecessary semicircles and line segments are erased. To construct the lower half of the repeat unit, replicate the procedure previously described and illustrated by **Figures 6b – 7e,** thereby taking advantage of the two-fold rotational symmetry of the repeat unit. The completed repeat unit design is shown in **Figure 7f**.

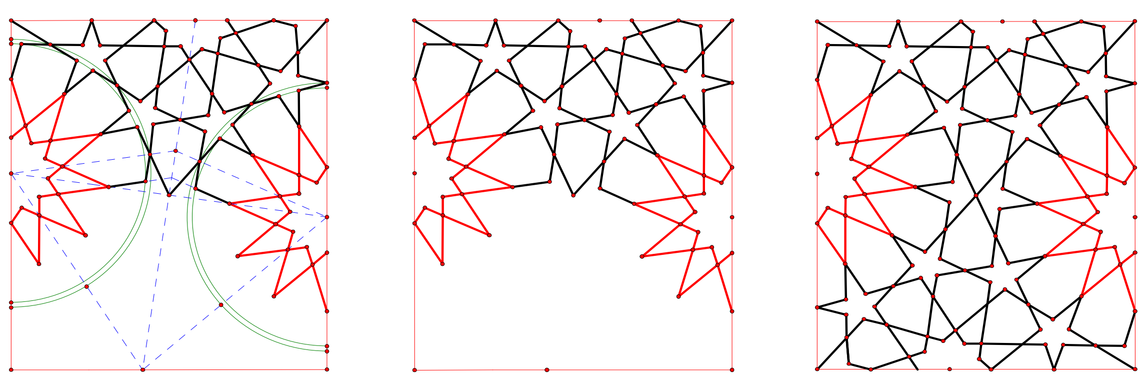

Figure 7d. **Figure 7e.** **Figure 7f.**

A colored rendition of sixteen copies of the repeat unit, replicated by reflection across the rectangle's edges, is provided in **Figure 8** on the next page. The symmetry of the pattern is unusual, with obvious vertical reflection mirrors bisecting the large eleven-stars, and horizontal reflection mirrors bisecting the smaller nine-stars. But, the 180 degree rotation centers at the corners and the center of the repeat units induce horizontal and vertical glide-reflections as well. The symmetry group of the pattern is *cmm*.

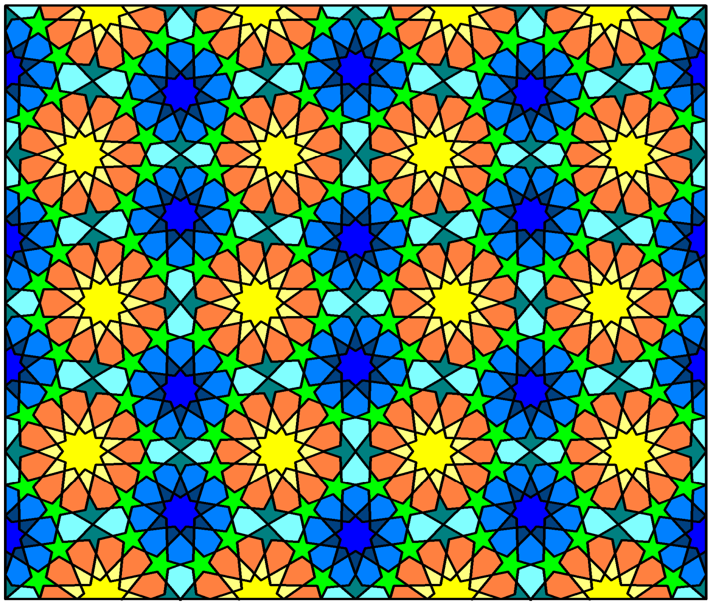

Figure 8. *Sixteen copies of the repeat unit, colored and replicated by reflection across the rectangle's edges*

Discussion

CN42 is the only idealized recorded pattern in the *Topkapı Scroll* containing eleven-pointed star polygons; there are no other such recorded pattern templates of which the author is aware. There are no eleven-pointed star polygons to be found in two rich published sources of Arabic geometric designs: Bourgoin's manual of 190 Islamic patterns, *Arabic Geometrical Pattern and Design* [5], and David Wade's collection of over 4000 images, now online [6]. Clearly CN42 is a very rare pattern.

Acknowledgements

The author is indebted to the reviewers for their useful comments and suggestions for improvement.

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