Mathematics Is Art

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Abstract

This paper gives a few personal examples of how our mathematics and art have inspired and interacted with each other. We posit that pursuing both the mathematical and artistic angles of any problem is both more productive and more fun, leading to new interdisciplinary collaborations.

Introduction

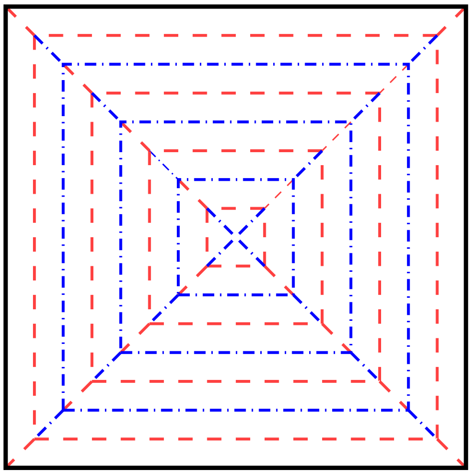
We come from two different backgrounds: Marty's first love was visual arts, while Erik's was mathematics. Over the years, we have learned extensively from each other, and worked collaboratively in both fields. Lately we have found the two fields to be converging more and more in our minds. No longer do we have separate art projects and mathematics projects: many of our projects have both artistic and mathematical angles, and we pursue both.

We find this approach to have several advantages. First, the art and mathematics inspire each other: building sculpture inspires new insights into the mathematics, and mathematical understanding inspires new sculpture. Second, it is harder to get stuck: if the mathematics becomes too difficult to solve, we can switch to illustrating the difficulty visually, and if a sculpture becomes too difficult to build, we can switch to developing a basic mathematical understanding of the structure to be built.

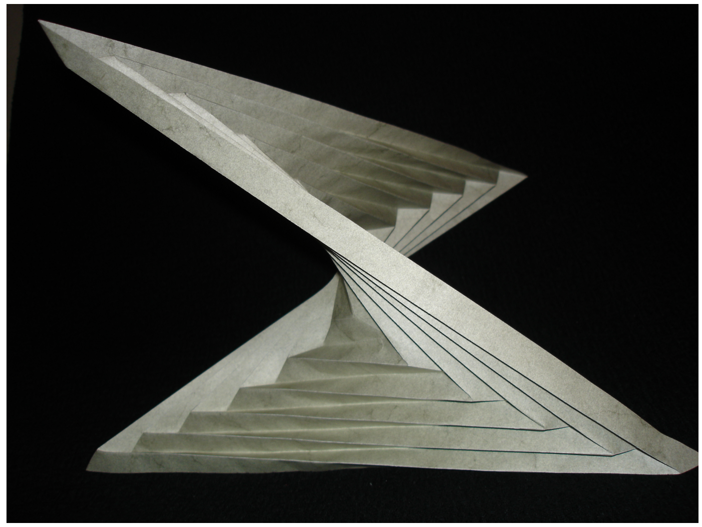
This paper gives a few personal examples of our projects that span both mathematics and art, and how we switched back and forth between the two fields. The first main example, pleated origami, has led to strong sculpture as well as interesting mathematics. The second main example, hinged dissections, has led to strong mathematics as well as interesting design. Both stories span a period of around ten years, with many small steps along the way in both art and mathematics.

Pleated Origami

Our first adventure in mathematical sculpture [7] appeared back at the second BRIDGES (1999), in a paper with Anna Lubiw (then our advisor). We were fascinated by a known geometric origami model, the "pleated hyperbolic paraboloid" or *hypar* shown in Figure 1. The folder makes a simple crease pattern—concentric squares and diagonals alternating mountain and valley—and then the model almost folds itself into a striking saddle surface. To turn these basic forms into more intricate structures, we designed an algorithm for generating sculptures composed of several hypars, automatically determined from a given input polyhedron. Figure 2 shows one example, which the algorithm produces when given a cube as input. Thus our study of pleating began in the artistic context, using mathematics (specifically algorithms) as a tool to design new sculpture.

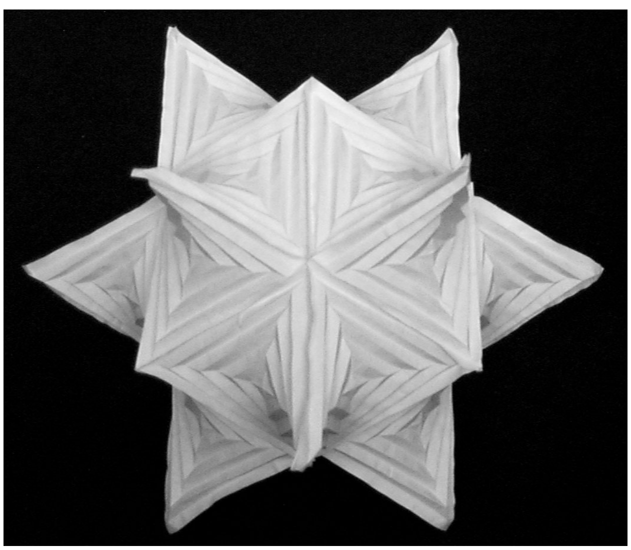


(a) *Hypar crease pattern.*



(b) *Folded hypar. [Photo by Jenna Fizel.]*

Figure 1: *Pleated hyperbolic paraboloid (hypar).*



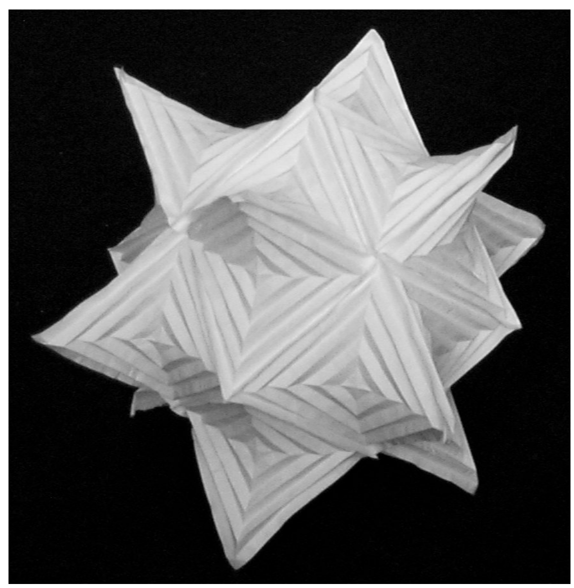


Figure 2: $24 = 6 \cdot 4$ *hypars arranged in the structure of a cube [7].*

Still intrigued by the self-folding nature of the hypar, in 2004–2006 we explored computer simulation of the material properties of paper. With Jenna Fizel (then an architecture student at MIT) and John Ochsendorf (an architecture professor at MIT), we succeeded in reproducing the physical form of hypars using a computer simulation, as shown in Figure 3. This simulation captures just two aspects of paper: the usual mathematical restriction that paper cannot stretch, and a physical property not studied in origami mathematics, the material elasticity of paper. The latter property is that paper tries to reach its memory state—flat for uncreased paper, and a particular fold angle at creases (depending on how hard the crease was made)—and searches for an equilibrium among these forces. This elasticity is the driving force that makes the hypar naturally take its saddle shape. Our simulation could even capture composite forms such as the 24-hypar representation of the cube, as shown on the right of Figure 3.

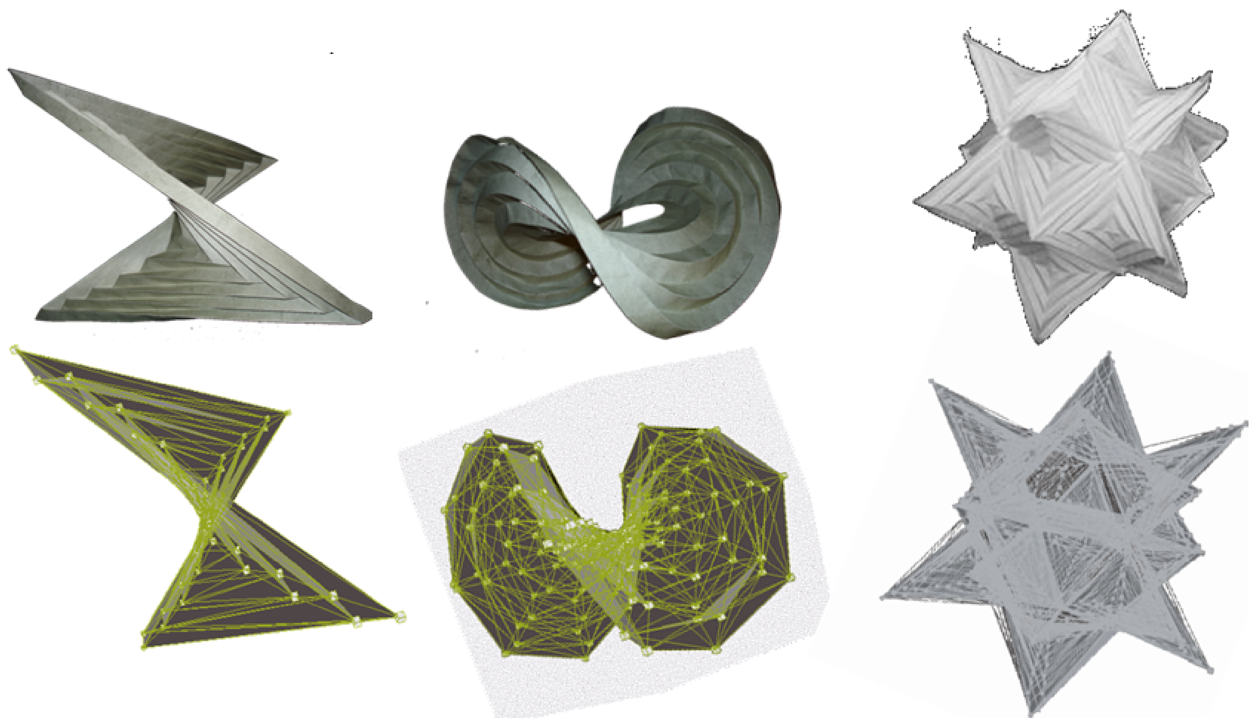


Figure 3: *Simulating self-folding origami. Top: Physical models of paper. Bottom: Analogous simulated models. Left to right: square, circle, and "cube" from Figure 2.*

Thus we turned to mathematics and science to better understand the artistic forms that we were working with. Armed with a computational tool for simulating these forms, we returned to the sculpture side, this time with the goal of illustrating that we can virtually simulate paper folding. Together with Jenna Fizel, we built the sculpture shown in Figure 4, which is a physical realization of a virtual simulation of a physical piece of paper, namely, a hexagon pleated with concentric hexagons and diagonals. This sculpture consists of aluminum rods, representing the creases of the form, and balls that are 3D printed to have holes at the angles found by our simulation. Approximately 1 meter in diameter, we view this construction as a model for a larger, climbable lawn sculpture.

Recently we discovered two surprising facts about the hypar origami model. First, the first appearance of the model is much older than we thought, appearing at the Bauhaus in the late 1920s [2]. Second, together with Vi Hart, Greg Price, and Tomohiro Tachi, we proved that the hypar does not actually exist [5]: it is impossible to fold a piece of paper using exactly the crease pattern of concentric squares plus diagonals (without stretching the paper). This discovery was particularly surprising given our extensive experience actually folding hypars. We had noticed that the paper tends to wrinkle slightly, but we assumed that was from imprecise folding, not a fundamental limitation of mathematical paper. It had also been unresolved mathematically whether a hypar really approximates a hyperbolic paraboloid (as its name suggests). Our result shows one reason why the shape was difficult to analyze for so long: it does not even exist!

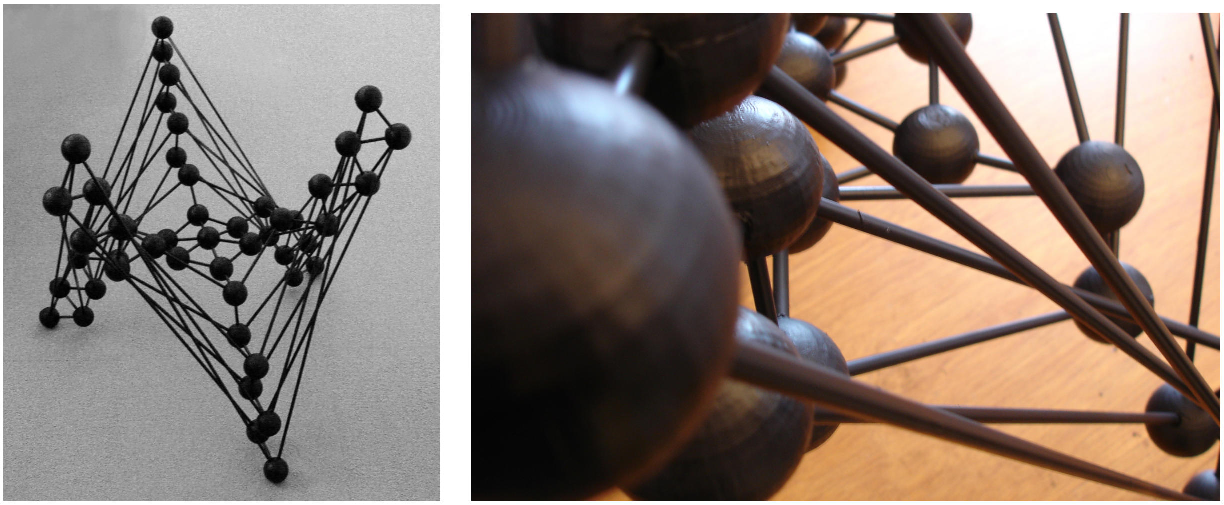


Figure 4: *Physical sculpture built from hexagonal hypar simulation.*

Curved Creases

Our adventure with pleated origami continued in the context of curved creases. Most origami uses straight creases, and in contrast to this origami, relatively little is known about curved creases and how they behave. A simple variation of the hypar, already shown in the middle of Figure 3, folds concentric circles instead of squares, this time with no diagonals but with a circular hole cut out of the center. This model also goes back to the Bauhaus in the late 1920s [2].

We have been experimenting with variations of this circular hypar for several years, as it naturally extends the hypar. On the mathematical side, the curved creases seem to behave quite differently from the straight creases of the hypar, as the resulting origami seems to actually exist [5]. A natural goal is curved-crease origami design: can we harness the power of these self-folding curved-crease forms to fold into desired 3D surfaces? How can we control the equilibrium form, and what (approximate) surfaces are even possible?

This mathematical challenge seems quite difficult, so we have so far focused this chapter of our adventure into the artistic realm. By experimenting with new forms and the effects of small variations, we simultaneously encounter new sculpture and get a better handle on how these forms behave mathematically. We hope one day to have a complete understanding of how to control the equilibrium form through a pleated crease pattern.

Our first main series, shown in Figure 5, is in the permanent collection of the Museum of Modern Art (MoMA) in New York. These pieces make one primary change to the circular hypar: they use a "circle" of total angle more than 360°. Given the hole in the center, the piece of paper is effectively a circular ramp that goes around two or three full circles before attaching to its starting point. Physically, such paper is formed by joining together multiple circles into one big cycle (without a topological Möbius twist). We find that this change to the paper drastically increases the geometric twist caused by the curved creases, and small variations can produce surprisingly different forms.

The MoMA pieces are initially scored with a laser cutter, then folded by hand. In 2009, we designed a new series folded entirely by hand for an exhibit at Art Cézar in Belgium. Figure 6 shows a few examples.

Figure 5: *"Computational Origami" (2008), in the permanent collection and on display at MoMA. Each of the three pieces is roughly 15 inches in diameter.*

Another way to vary the concentric pleating idea uses flat paper but noncircular curves. We started exploring some of these variations in 2003 with abhi shelat (then an MIT PhD student in computer science), including ellipses and partial circles. Most recently, with Duks Koschitz (an MIT PhD student in architecture), we have explored a wide variety of curves with different offset patterns [10]; see Figure 7 for two examples. Our primary mathematical goal is to understand what pleating "works" (produces an interesting 3D form). Along the way, we find interesting sculptural forms.

We are also interested in finding other materials that fold in a way similar to paper, with the motivation of making larger and/or stronger structures. Such materials would be useful for larger sculpture as well as more practical applications like furniture, buildings, and other large-scale constructions. One promising alternative is metal, which is difficult to fold, especially manually, but can produce striking results when successful.

Figure 6: *"Waves" (2009), exhibited at Art Cézar in Belgium. Each of the three pieces is roughly 15 inches in diameter. In reading order: Inner Wave, Splash, and Three Waves Meeting.*

Hinged Dissections

A hinged dissection is a chain of polygonal pieces, hinged together at vertices, that can fold into multiple desired polygons. Perhaps the most famous example, shown in Figure 8, is Dudeney's hinged dissection of an equilateral triangle into a square. Hinged dissections have been thoroughly studied and designed [9], but until recently lacked a general theory. In particular, the major open question that intrigued us was whether hinged dissections exist for any two polygons of the same area.

We started exploring hinged dissections from the mathematical perspective in 1999 [4]. Initially inspired by a web discussion between David Eppstein and Erich Friedman, together we found hinged dissections for "polyforms": shapes made out of n copies of a common shape. The most popular example of polyforms is polyominoes, of Tetris fame, made up of n copies of a unit square joined edge to edge. Our result shows that one hinged dissection can fold into all (exponentially many) polyforms for any desired base shape (squares, equilateral triangles, right isosceles triangles, hexagons, etc.) and any desired value of n. The hinged dissection is also quite simple, dividing each copy of the base shape into a few pieces.

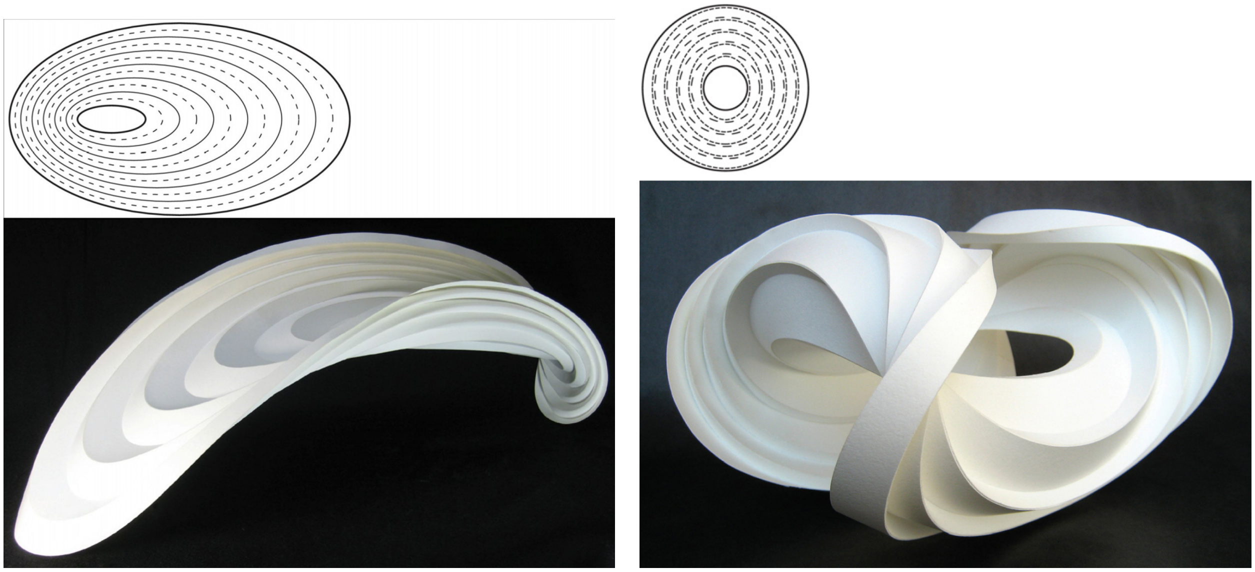


Figure 7: *Two curved-crease models from [10]. Top: Crease patterns. Bottom: Folded models.*

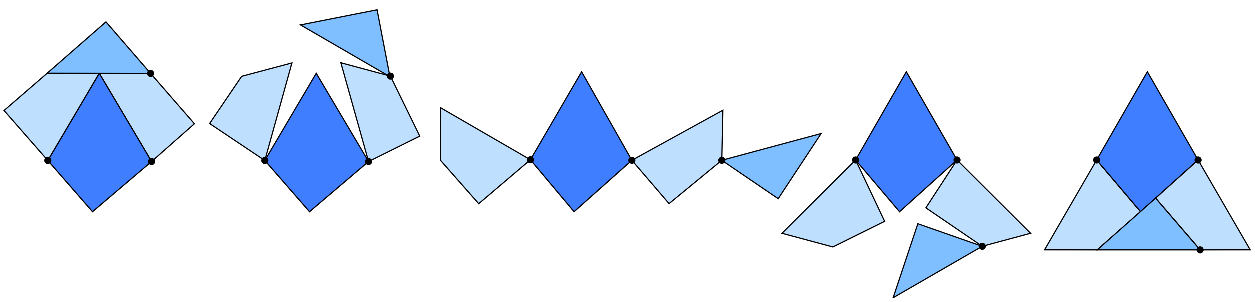


Figure 8: *Dudeney's hinged dissection from 1902.*

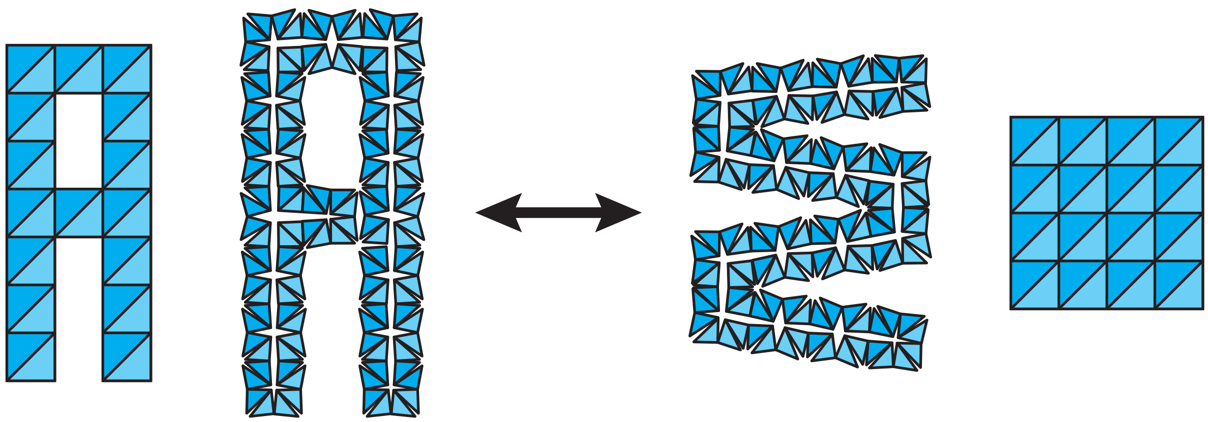
Later, Greg Frederickson joined in, and we found that one hinged dissection could fold into polyforms with different base shapes, such as both squares and triangles, by incorporating a hinged dissection similar to Figure 8.

Next we turned to the artistic side, this time in the context of design. On the one hand, we wanted to illustrate the generality of the mathematical result: one hinged dissection could fold into many many shapes. On the other hand, we have always been fascinated by font design. We combined these two desires by designing a font [3], shown in Figure 9, with the property that one hinged dissection can fold into every letter and digit of the font as well as a square. To achieve this property, we needed every letter and digit to have the same area and to be a polyform, in this case with the base shape a right isosceles triangle. The smallest interesting size for the square seemed to be 4×4, resulting in $32 = 4 \cdot 4 \cdot 2$ copies of the base shape and $128 = 32 \cdot 4$ pieces in the hinged dissection.

Returning to mathematics, we found that the hinged-dissection technique of [4] could be applied even more generally. Together with Jeff Lindy and Diane Souvaine, we proved that one hinged dissection could fold into all 3D polyforms, or "polypolyhedra", where the base shape is now a polyhedron such as a cube [6].

While writing down this result, we were contacted by Laurie Palmer, a professor in sculpture from the

(a) *Alphabet designed to have a hinged dissection.*



(b) *Shaping the hinged dissection into the letter A and the square.*

Figure 9: *Hinged dissection of the alphabet [3].*

School of the Art Institute of Chicago who was then a Radcliffe Institute Fellow. She wanted to build an interactive sculpture that the viewer could fundamentally change in shape, from a one-dimensional line to a two-dimensional plane to a three-dimensional solid. She already had around 1,000 identical wooden blocks and piano hinges for connecting them together. The striking similarity to the paper we were working on at that moment was a pleasant surprise. We noticed that the hinged dissection for the base shape of a cube generalized to any parallelepiped, in particular her rectangular blocks. We described the (regular) pattern in which she had to hinge the blocks in order to guarantee universal folding, and she built the installation, *The Helium Stockpile*, which exhibited at Radcliffe in April 2004; see Figure 10. Excited about our experience with the collaboration, we wrote a paper together about it [8].

Back to the mathematical realm, last year we finally conquered the general hinged-dissection problem, proving that hinged dissections exist for any desired set of polygons of equal area [1]. The solution required key new insights made by four students: Timothy Abbott (MIT), Zachary Abel (Harvard), David Charlton

Figure 10: *Laurie Palmer's The Helium Stockpile, April 2004 [8].*

(Boston University), and Scott Kominers (Harvard). After nearly ten years of pursuing this problem from artistic and mathematical angles, it is exciting to see the main mathematical problem finally answered. Of course, now the challenge is to design interesting art using the mathematical theory.

Glass

Marty's first artistic endeavor was glass blowing, and returning to those roots, both authors are now active glass blowers at MIT. A recent project with Amy Nichols (MIT glass blower and PhD student in biological engineering) combines our interests in folding and hot glass by attempting to fold glass. A unique property of hot glass is that we cannot touch it (being at around 1400°F), which requires a fundamental shift from the usual approach to folding origami. We chose to embrace the use of gravity as a primary force in glass blowing. To help gravity fold the glass in interesting ways, we fused the initial piece of glass from two chemical mixtures—stiff white glass and soft black glass—and built a simple annealing chamber providing obstacles during the folding process. After this setup, the folding was done entirely by gravity. Figure 11 shows our first sculpture made by this approach, which we hope will lead to many more interesting forms.

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Figure 11: *A folded glass sculpture.*

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