The Catenary
Art, Architecture, History, and Mathematics

Gail Kaplan
Department of Mathematics
Towson University
Towson, Maryland 21252, USA
gkaplan@towson.edu

Abstract

This paper describes how to use the catenary curve to enable students to see and appreciate connections between mathematics and other disciplines, including history, art, and architecture. Students gain a more in depth understanding of the mathematics through the links to other fields.

I.   Introduction   A catenary is the name for a curve that occurs naturally when a chain of uniform density is allowed to hang. The word itself is derived from the Latin, *catena*, which means chain.

II.   Mathematics   In applications mathematicians are often interested in finding a function to represent a particular shape or situation. Consider the following three curves, all with a similar appearance.
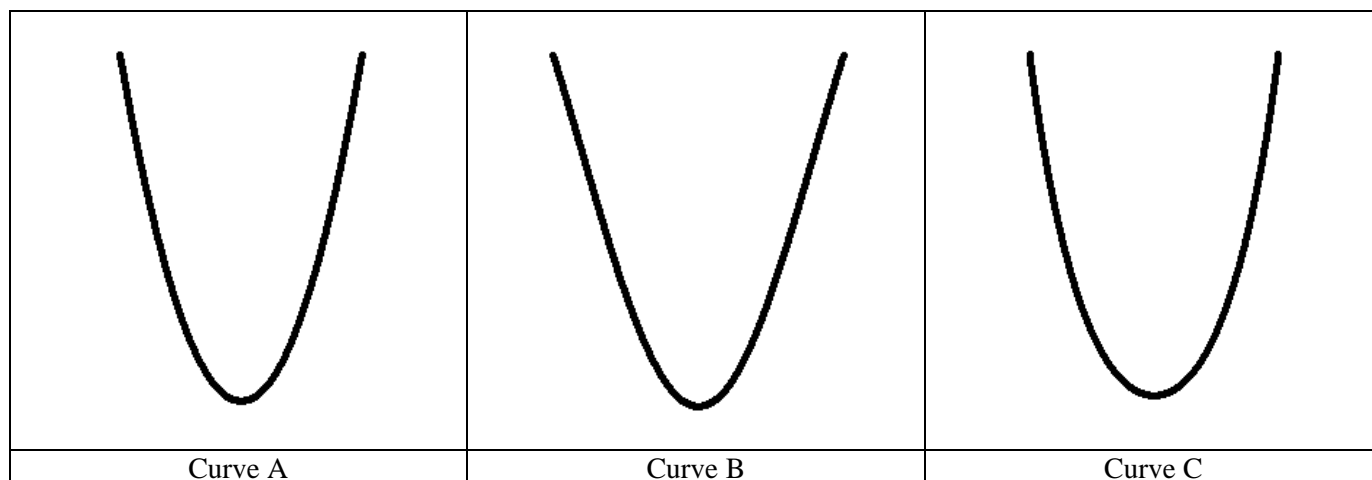

| Curve A | Curve B | Curve C |

Figure 1

Curve A is a parabola, the graph of a quadratic equation generally introduced in a high school algebra course. The curve shown is a portion of the graph of $p(x) = x^2$. Curve B is part of the graph of a trigonometric sine curve, generally studied in a high school precalculus course. The curve shown is a portion of the graph of $f(x) = \cos x$. Curve C is a portion of a hyperbolic cosine curve, also known as a catenary curve. Visually, we see that the curves appear to have the same basic shape. The basic equation for a parabola is $p(x) = x^2$. The general form for the hyperbolic cosine is $c(x) = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}} \right)$. Since $x^2 = (-x)^2$, $p(x) = p(-x)$. The trigonometric identity $\cos(x) = \cos(-x)$ implies that $f(x) = f(-x)$. Since
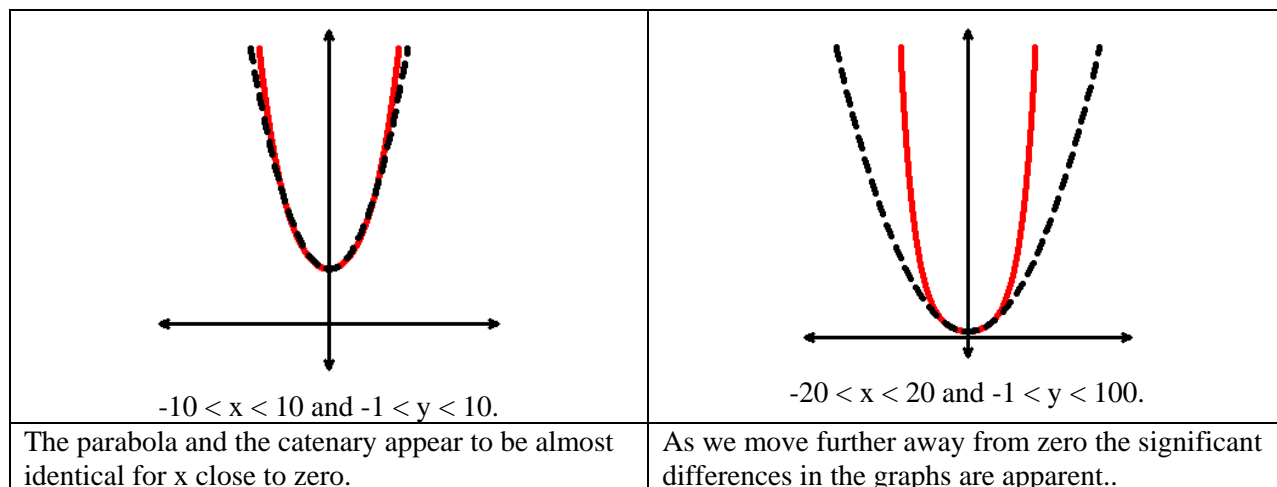
$c(-x) = \dfrac{a}{2}\left(e^{\frac{-x}{a}} + e^{\frac{-(-x)}{a}}\right) = \dfrac{a}{2}\left(e^{\frac{-x}{a}} + e^{\frac{x}{a}}\right) = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right) = c(x)$, the catenary is an even function. Thus, all three curves share this property. By using simple transformations of these functions we can superimpose one on another. In the following diagram we readily see the similarity of the shapes.

The solid line represents the hyperbolic cosine, the dotted line represents the cosine function, and the dashed line represents the parabola.
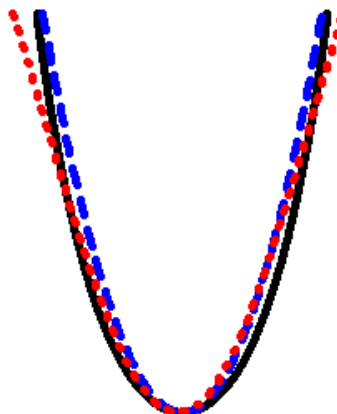
Figure 2

For relatively small values of x the parabola and catenary appear quite similar, but as x increases in size the difference is rather dramatic. Consider the following images of the graphs of the functions $c(x) = e^{\frac{x}{2}} + e^{\frac{-x}{2}}$ and $p(x) = .3x^2 + 2$. Notice how similar the shapes appear for x close to zero. Yet as we move further from zero, the two functions begin to vary dramatically due to the exponential.

-10 < x < 10 and -1 < y < 10.	-20 < x < 20 and -1 < y < 100.
The parabola and the catenary appear to be almost identical for x close to zero.	As we move further away from zero the significant differences in the graphs are apparent..

Figure 3

Let us look more closely at the graph of the catenary, $c(x) = e^{\frac{x}{2}} + e^{\frac{-x}{2}}$. Notice that $\lim\limits_{x \to \infty} e^{\frac{-x}{2}} = 0$ and

$\lim\limits_{x \to -\infty} e^{\frac{x}{2}} = 0$. This tells us that the catenary is asymptotic to $y = e^{\frac{x}{2}}$ as x approaches positive infinity and

is asymptotic to $y = e^{\frac{-x}{2}}$ as x approaches negative infinity. These asymptotes are illustrated in the following graph. Note that neither the parabola nor the sine function has a similar property.
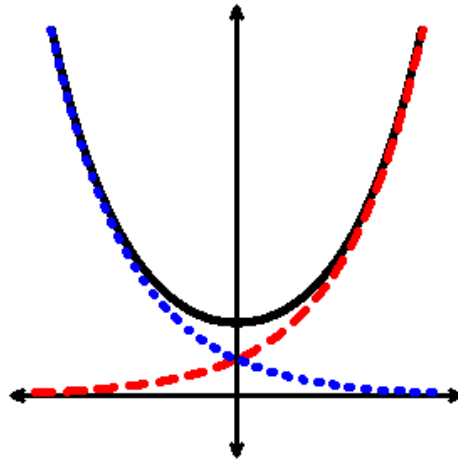

Figure 4

III.   History   "It's A Bird, It's A Plane, It's Superman!" or "It's A Parabola, It's A Parabola, It's a Catenary!" The radical views of Galileo, the brilliant Italian astronomer born in 1564, stimulated major controversy in his time. The Catholic church viewed the earth as the center of the universe, but Galileo correctly determined that the earth revolves around the sun. This gifted man also studied projectile motion, the path of an object thrown into the air at an angle, like a soccer ball or a baseball. Galileo was able to show that these objects follow a parabolic path. However, although Galileo was the first to study a catenary, he mistakenly identified the shape as a parabola. He incorrectly believed that a hanging rope created the shape of a parabola. If a cable is hung from two vertical supports, it actually forms a catenary. The figure below illustrates a cable hung from two posts. It forms a catenary. This can be shown by solving a differential equation representing the forces on each point of the curve.
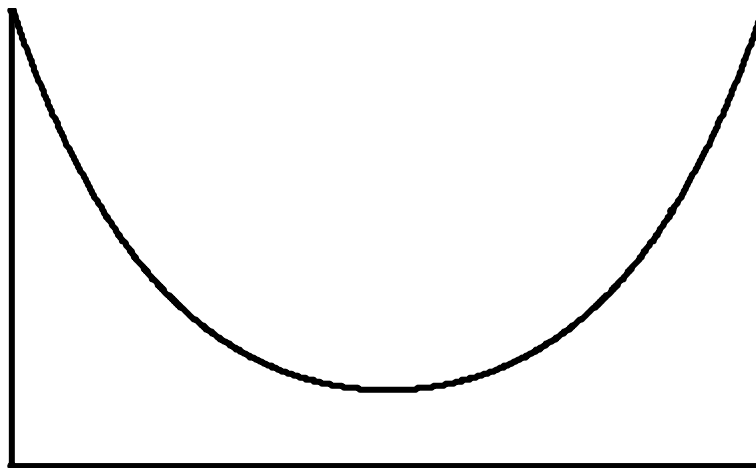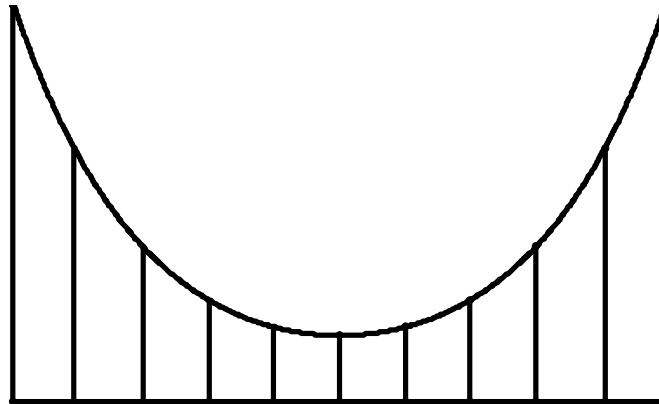

Figure 5

If we imagine that this is part of a bridge and add vertical cables to support the horizontal span, we have the following illustration. The vertical cables are now supporting the roadway of the bridge. When this construction is made, the curve will be a parabola. This, too, can be shown by solving a differential

49

equation representing the forces on each point of the curve. Thus, for a suspension type bridge Galileo was correct to conclude that the cable forms a parabolic shape.
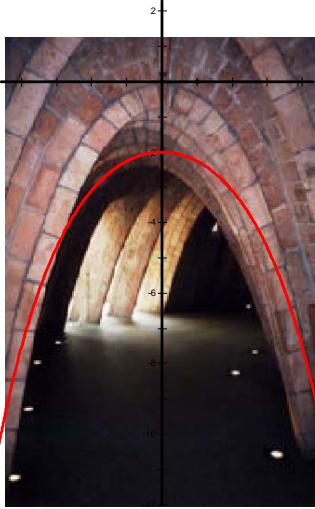

Figure 6

In 1669 Joachim Jungius, a German mathematician interested in mathematics as a means to describe physical science, showed that a catenary shape is not a parabola.[1] Responding to a challenge from Jakob Bernoulli in 1690, his contemporaries Christiaan Huygens, Johann Bernoulli, and Leibniz correctly and independently described the curve mathematically in 1691. Huygens was the first to actually use the word catenary; this was in a letter written to Leibnitz on November 18, 1690. Jakob Bernoulli called the curve vélaire, a French word related to the nautical meaning of sail.

IV.   Architecture   Antonio Gaudi, the creative Spanish artist said "There are no straight lines or sharp corners in nature." Gaudi designed a unique apartment building in Barcelona called Casa Mila. the building is utterly amazing and significant portions are open for exploration. On the top level there are beautiful arches as shown in figure. Although they appear to be parabolic in shape, Gaudi's creations are actually the shape of a catenary.

Figure 7

Let us now begin to examine the curve mathematically by looking at a slice of a single arch.

The general form for the equation of a catenary is $c(x) = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$. We begin by plotting this equation for a = 2, $c(x) = \left(e^{\frac{x}{2}} + e^{\frac{-x}{2}}\right)$	We adjust the equation to transform the catenary so that it approximates the arch. In the first step we plot $-c(x) = -\left(e^{\frac{x}{2}} + e^{\frac{-x}{2}}\right)$ to reflect the curve over the x-axis.	Next, we translate and stretch the function. In the figure below we see that the equation $-c(x) = -1.7\left(e^{\frac{x}{2}} + e^{\frac{-x}{2}}\right) + 2.5$ is an excellent approximation to our picture of the arch.

Figure 8

In Minneapolis, Minnesota the Marquette Plaza designed by Gunnar Birkerts, is a fabulous example of a building where the centenary is not only an integral part of the exterior design, but also provides a very unique visual image.

Figure 9

The stunning Kingdom Towers designed by Ellerbe Becket & Omrania is located in Riyadh, a city in Saudi Arabia. The incorporation of the out of the ordinary mathematical shape is part of the breathtaking architectural design.

Figure 10

Eero Saarinen and Associates designed the Dulles International Airport which also incorporates the catenary design in the eye catching shape.
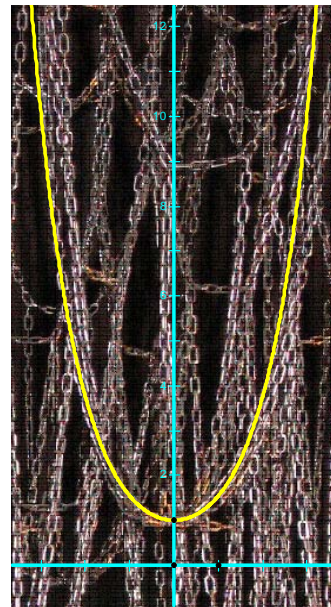

Figure 11

V.   Art   We now examine two examples of sculpture with a catenary shape. The famous Gateway Arch in St. Louis, Missouri is one of the few sculptures where a mathematical equation for the shape, $y = 68.8\cosh(0.01x - 1)$, is embedded into the structure. In the first picture we see the beauty and simplicity of the sculpture. In the second picture, the graph of the function $y = 68.8\cosh(0.01x - 1)$ is placed on top of the actual arch. The formula appears to be correct!

Figure 12

In Casa Mila, the apartment building designed by Gaudi there is a fabulous sculpture of hanging chains on the top level. Since chains naturally form the shape of a catenary, this displays many catenaries in one sculpture. In the following diagram, the graph of a catenary is placed on top of one of the chains. We see that the equation describes the precise shape of the hanging chain.

Figure 13

VI.   Engineering   The inverted catenary is an ideal shape for arches as its physical properties ensure that the arch supports itself. Potters use catenary kilns for this reason. In a kiln the extreme differences in temperature create problems in the integrity of the structure. Kilns built in circular shapes need additional structure to preserve their integrity; however, kilns built in the shape of a catenary do not have the same issue. It is interesting to note that although it is possible to find equations to use to build catenary kilns, the common advice is to hang a chain, trace it, invert it, and then use the sketch for your blueprint. That way the mathematically anxious kiln builder is more comfortable since the mathematics is hidden! The following illustration is a catenary kiln in construction.

Figure 14

In the following photograph of the Hohenzollern Bridge in Cologne, Germany we see an inverted arch that is in the shape of a catenary, creating a stable structure. The suspension cables support the bridge.

Figure 15

VI.   Conclusion   The catenary function is a wonderful example of how mathematics, art, architecture, and history overlap. It provides an opportunity to motivate students to explore an idea using an interdisciplinary approach.

References
The MacTutor History of Mathematics Archive:
http://www-groups.dcs.stand.ac.uk/~history/Biographies/Jungius.html

http://whistleralley.com/hanging/hanging.htm

Encyclopedia Britannica Online
http://www.britannica.com/eb/art-68186/Catenary-and-exponential-functions-Any-nonelastic-uniform-cable-held-at

Wikipedia, the free encyclopedia
http://upload.wikimedia.org/wikipedia/commons/8/87/Hohenzollernbr%C3%BCcke-in-der-D%C3%A4mmerung-K%C3%B6ln.jpg
http://en.wikipedia.org/wiki/Image:CatenaryKilnConstruction06025.JPG
http://upload.wikimedia.org/wikipedia/en/3/35/Kingdom_Tower.JPG
http://upload.wikimedia.org/wikipedia/commons/a/a8/Marquette_Plaza.jpg
http://upload.wikimedia.org/wikipedia/en/6/68/Dulles_Airport_Terminal.JPG

Footnotes
1.   http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Jungius.html
2.   http://www.2dcurves.com/exponential/exponentialhc.html