Some Regular Toroids

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Abstract

We present examples of polyhedra homeomorphic to a torus such that all of their faces are planar polygons with the same number of sides and all of their vertices are incident to the same number of edges. Neither the faces nor the polyhedra are self-intersecting. We call them regular toroids. Because of their combinatorial regularity, these structures may be attractive to people who find beauty in geometric patterns.

It is easy to see that there are three classes of regular toroids. These classes are denoted by the Schläfli symbol $\{p, q\}$ which means that p edges belong to a face and q to a vertex. Thus the classes are: {3, 6}, {4, 4} and {6, 3}. This corresponds to the fact that one can tessellate the plane by regular polygons in the same manner. Within each class, it is easy to construct representatives with sufficiently large face number. A sufficient condition for this number is that it has at least two factors each of which is greater than two.

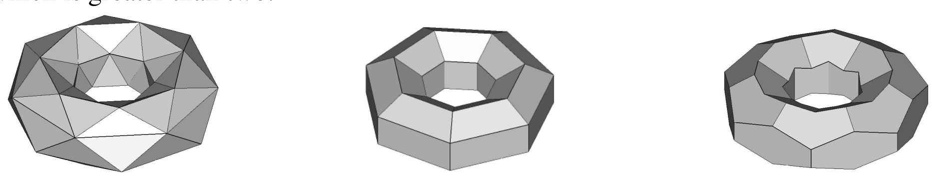

Figure 1*: Representatives of the class {3, 6}, {4, 4} and {6, 3}, with (V, E, F) = (42, 126, 84), (42, 84, 42), (84, 126, 42), respectively*

The question is whether these classes have representatives that do not satisfy this condition. If yes, which of them has the minimal number of vertices or faces?

The vertex-minimal representative of the class {3, 6} is the Császár polyhedron having $(V, E, F) = (14, 21, 7)$. See [1]. We can show that in the class {3, 6} there is a regular toroid for each $7 \le V \le 12$ such that it is non-intersecting. However, for these smaller V it is difficult to find vertex coordinates that meet this requirement.

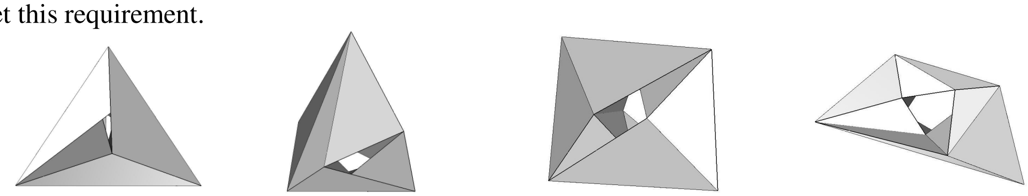

Figure 2: *The polyhedra of the class {3, 6} with V = 7, 8, 10 and 11*

In the class {4, 4}, for the polyhedron with minimal number of faces and vertices we have $(V, E, F) = (9, 18, 9)$. For the next polyhedron $(V, E, F) = (12, 24, 12)$. The author does not know if there is a toroid with the face number not satisfying the sufficient condition mentioned in the first paragraph, for example, with $F = 10, 11, 13, 14, \ldots$

In the class {6, 3} the minimal number of faces is 7. This polyhedron was discovered by the author, and it was Martin Gardner who first called it the *Szilassi polyhedron* [2]. A metallic sculpture of it is exhibited in the mathematical museum of Fermat's birth house.

One can easily construct polyhedra in this class with face numbers 9, 12, 15, …; in general, this number is of the form $a * b$, $a \ge 3$, $b \ge 3$ [3]. Since 1988 we have known a toroid with 8 hexagons (see [4]); however, this is an overarching polyhedron, that is, for each of its faces there is another face such that they have two edges in common.

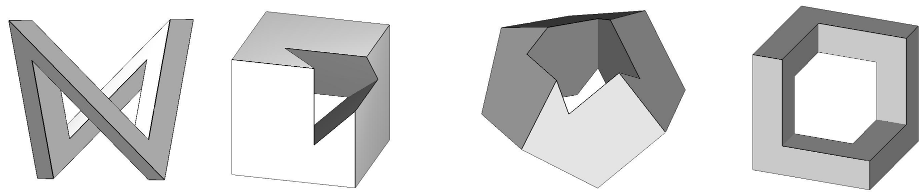

Figure3: *Polyhedra in the class {6, 3}. Leftmost: an overarching polyhedron with F= 8; for the next one F= 9; the last two examples, although both has 12 faces, are combinatorially non-isomorphic.*

We can show that **there exist regular toroids with 8, 10 and 11 faces in the class {6, 3}.** It is an open problem whether there exists a regular toroids with 13 or 14 faces. The problem does not become easier with an increasing number of faces, unlike the case of {3, 6} toroids.

We obtained (in 1977) the toroid with 7 hexagons from the Császár polyhedron by applying to its faces, vertices and edges a reciprocation with respect to a sphere. Since the Császár polyhedron is necessarily non-convex, and the polarity is a projective geometric transformation, in the dual figure self-intersecting faces were produced. To explore these undesirable self-intersections, we used a computer-aided procedure. By varying the vertex positions of these 7-vertex polyhedra we succeeded in eliminating all self-intersections of any faces. These investigations, required several months with the computers of that time; now they can be performed within minutes. A program suitable for this purpose is *Euler 3D* (http://www.euler3d.hu/). This has made possible to find polyhedra in the class {6,3} that were unknown till now. If we are able to find coordinates for a polyhedron with a given number of faces and a given combinatorial structure, such that its dual obtained by a suitable reciprocation is free of self-intersections, then we can further fine-tune the coordinates with some ”aesthetic” goals in mind. For example, we could aim for some more symmetrical appearance. The polyhedra in Figure 4 with face number 8, 10 and 11 are respectively the duals of the polyhedra in Figure 2 obtained by reciprocation with respect to a suitable sphere.

Since the vertices of the starting polyhedra of type {3,6} have rational coordinates, and in applying the reciprocation one has to solve systems of linear equations, we obtain toroids of type {6, 3} also with rational coordinates.

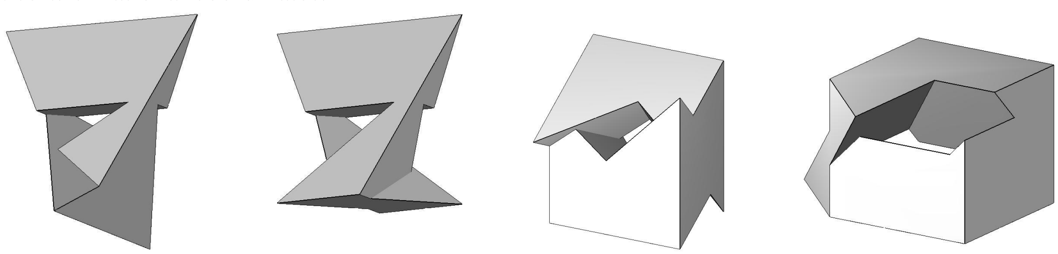

Figure 4: *The polyhedra with face numbers 7, 8, 10 and 11 of the class {6, 3}*

In the CDROM of the conference one can find the movable (Euler 3D) models of the polyhedra presented here; one can also read out from them the combinatorial structure and the coordinates of these polyhedra as well.

References

[1] Á.. Császár, *A polyhedron without diagonals*, Acta Sci. Math. (Szeged) **13**, pp.140–142. 1949-50

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[3] L. Szilassi, *Regular Toroids,* Structural Topology / Topologie structurale pp**.** 69–80, 1986, 13,.

[4] J. Schwörbel, *Die kombinatorisch regulären Tori.* Dipl. thesis, Siegen University, 1988