A Fractal Crystal Comprised of Cubes and Some Related Fractal Arrangements of other Platonic Solids

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Abstract

A simple iterative arrangement of cubes leads to a visually rich and complex fractal "crystal" with an overall regular-octahedron convex hull and infinitely many "facets". Each facet is essentially a Sierpinski triangle, and the vertex of a cube just touches the center of each triangular hole. This fractal crystal is constructed by starting with a first generation cube and placing a half-scale cube on the center of each face. The second-generation cubes have the same orientation as the first-generation cube. Third-generation cubes again scaled by half are placed on each unoccupied face of a second-generation cube. This process is continued *ad infinitum* to form the fractal crystal. Some related constructions created using other Platonic Solids are described as well.

Introduction

Classical fractals such as the Koch Snowflake and Sierpinski triangle [1] are created by iteratively applying a modification to simple geometric structures such as a line segment or an equilateral triangle. Similar constructions are possible in three dimensions; the best-known examples are probably the Sierpinski tetrahedron and Menger Sponge [1]. In this paper, we describe a particular family of fractal constructions created by iteratively arranging smaller copies of Platonic Solids on the faces of larger copies, where the starting point is a single first-generation solid. The fractal structures that result from such iterative arrangement of solids can exhibit unexpected features.

In the 1970's, Martin Gardner speculated in one of his columns that a 3-dimensional analog of the Koch Snowflake would have a fractal, crinkly surface [2]. William Gosper and Hans Morovec wrote a computer program to calculate what this structure would really look like, and found out that it actually forms a cube [3]. A particular orthographic projection of this structure is illustrated in the top half of Figure 1. After two generations, the vertices of the tetrahedra coincide with the vertices of the final cubic convex hull. Adding additional generations basically fills in the cube. This structure can be shown to have infinite surface area, yet the same volume as the cube. This structure was written about in Omni Magazine in the 1980's [4] and a student activity based on it is available [5]. Another way of looking at this structure is to consider the second generation version to be a stellated octahedron or stella octangula [6]. If each of the eight visible tetrahedra are then replaced with scaled-down stellated octahedra, etc., a similar figure is obtained. This structure contains octahedral cavities that get larger in number and smaller in size with each generation [6].

An analogous structure can be envisioned for cubic building blocks, where each face of the cube would have a 1/3 scale cube placed on it, thereby dividing the square face into 13 smaller squares. Each of these smaller squares can be modified in a similar fashion, as shown in the bottom half of Fig. 1.

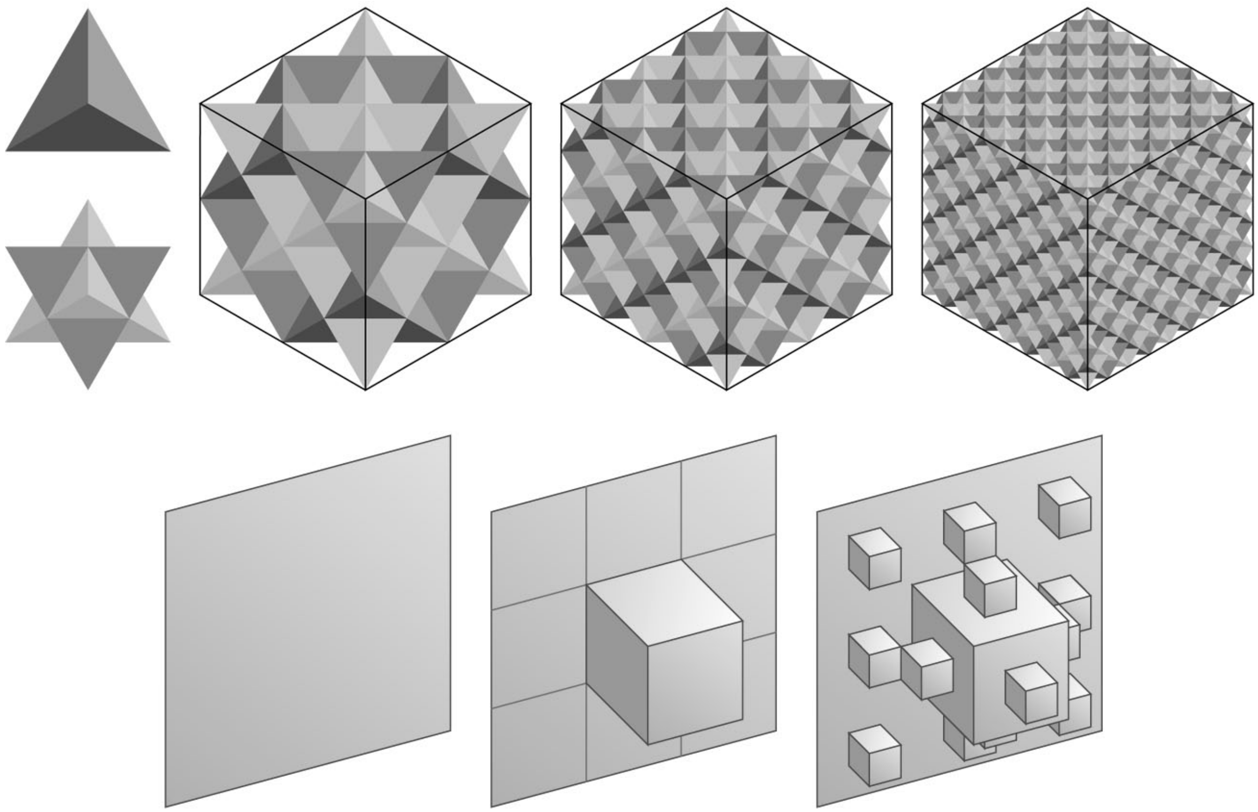

Figure 1: *Top: The fractal arrangements of tetrahedra described in [5], with the first two generations shown smaller, and the limiting cube shown in generations 3-5. Bottom: An analogous arrangement of cubes, where the first three generations of one face of the starting cube are shown.*

The Koch Snowflake and the structures in Figure 1 share the property that each individual line segment or simple-polygon surface element is infinitesimally small in the limit of an infinite number of iterations. In this paper, some less conventional fractal arrangements of polyhedra are described, in which portions of the polyhedra are left unmodified at each generation. From the standpoint of mathematics, these structures might be considered less pure, but from the standpoint of mathematical art, the presence of features with a range of sizes can create more visual interest. Due to the fact that these are simple and straightforward arrangements of polyhedra, some and possibly all of them have been previously discovered. References are provided to examples of them in the literature of which the authors are aware.

A Fractal Crystal Comprised of Cubes

We have previously described a wide variety of 2-dimensional fractal tilings [7]. While most of these are edge-to-edge arrangements of irregular polygons, fractal tilings of squares and other regular polygons have been explored as well. A fractal tiling of squares is shown in Figure 2. The starting point is a single first-generation square. Second-generation squares scaled by 0.5 are arranged adjacent to and centered on each of the four edges of the starting square. Third-generation squares are then arranged around second-generation squares unless the edge is already occupied by the first-generation square. This process is continued *ad infinitum* to yield the fractal structure of Figure 2. This arrangement fills in a larger square except for a set of measure zero consisting of points in the fractal arrangement partially indicated in Figure 1 by white channels.

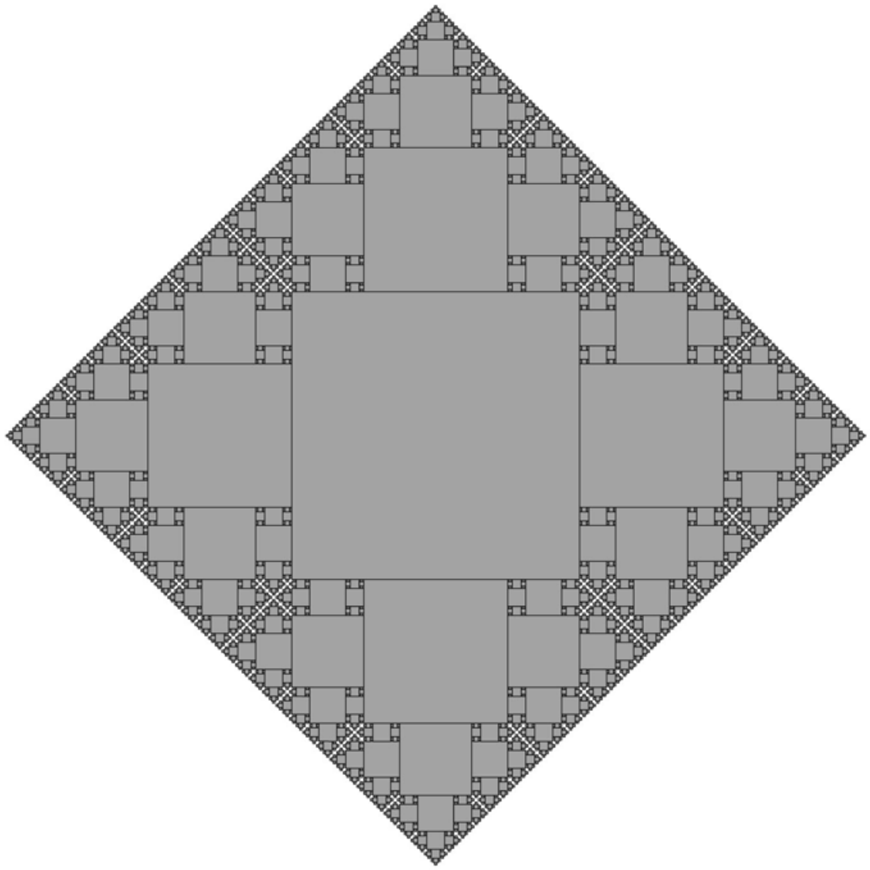

Figure 2: *A fractal arrangement of squares that is a cross-section of the fractal crystal shown in Figures 3 and 4.*

Using this arrangement of squares as a starting point, it is a straightforward extension to imagine an analogous arrangement of cubes. Visualizing this three-dimensional structure is not straightforward, however. It is clear that the arrangement of squares in Figure 2 would be an equatorial cross-section of the arrangement of cubes, and that there would be three such slices, each orthogonal to the other two. An orthographic projection of the arrangement carried through six generations of cubes is shown in Figure 3 [8]. After drawing 4-5 generations, it becomes apparent that the convex hull of this structure is a regular octahedron, which is, not coincidentally, the dual of the cube. After 6-7 generations, it becomes apparent that each of the faces of this octahedron contains a series of triangular holes in the configuration of a Sierpinski triangle. A single larger cube just touches the center of each equilateral triangle. We have found the same arrangement of cubes, through fewer generations, in a couple of other places [9]. However, the Sierpinski triangle feature, which is what makes this particular arrangement of cubes so compelling, does not appear to have been recognized before.

A VRML description of this arrangement of cubes was generated through 11 generations and used to control a zCorp 3D color printer for creation of a physical model [10]. The model, which took 36 hours to write, measures approximately 8” along each edge of the overall octahedron. Red was chosen as the color for the first-generation cube, with gradual color changes with each successive generation up to the eleventh, which is blue. Photographs of the physical model are shown in Figure 4. The complexity of the object, combined with the simplicity of the arrangement of cubes and the overall octahedron convex hull, make it interesting both as a mathematical and as a sculptural object. While the computer description includes cubes up to the eleventh generation, the physical limitations of the printing process prevent individual cubes beyond the seventh generation from being discernible.

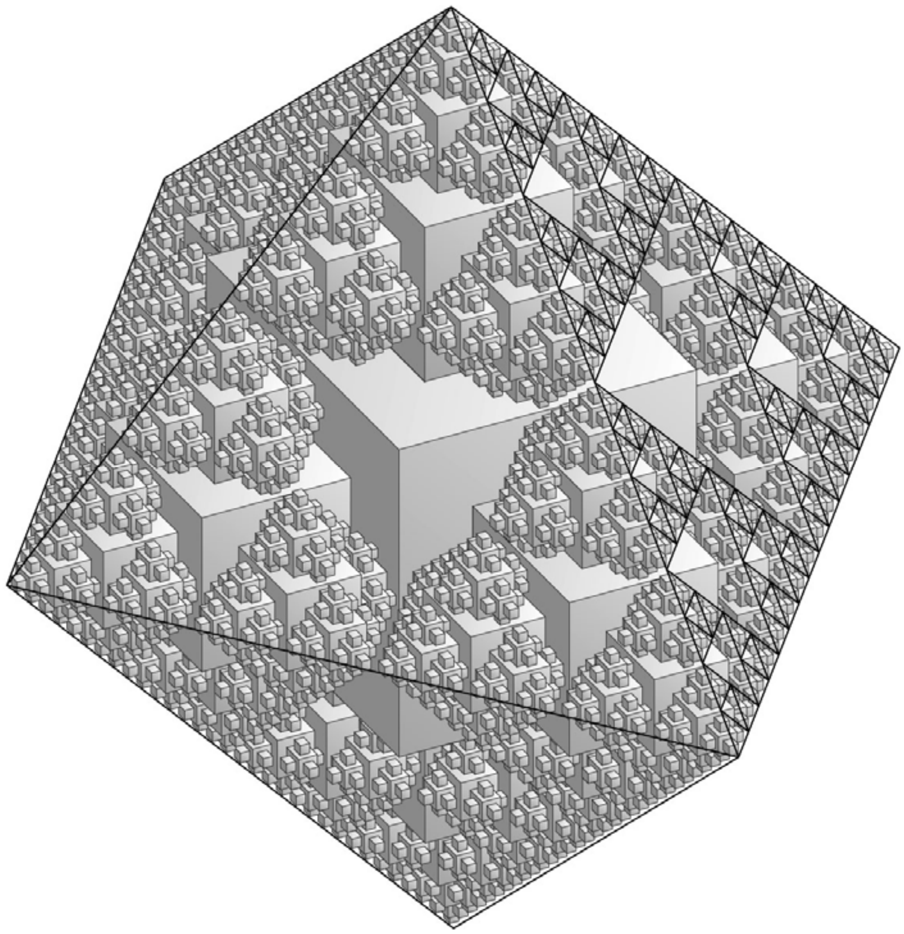

Figure 3: *Orthographic projection of a fractal arrangement of cubes that is closely related to the arrangement of squares shown in Figure 1. The octahedral convex hull is shown in black lines, along with a Sierpinski triangle on the upper-right face.*

This object is aptly described as being a “crystal”. One can think of the development of the structure with added generations as being analogous to crystal growth. Growth occurs most rapidly along directions normal to the faces of the starting cube, resulting in triangular facets. In natural crystals, faceting along crystallographic planes is a result of a growth process that is influenced by factors such as the relative surface energy of different planes.

Other Platonic Solids

Tetrahedra. Analogous fractal arrangements can be formed using the other Platonic Solids. The tetrahedra, octahedra, and dodecahedra cases are described in this section, one goal being determining which ones generate interesting sculptural forms. The tetrahedron is distinct from the other Platonic

Solids in that it lacks opposing parallel faces. This means the orientation of the various tetrahedra cannot all be the same, but rather must alternate between two orientations from one generation to the next. Each scaled-down tetrahedron occupies the center quarter of an equilateral-triangle face of a larger tetrahedron. In analogy to the cubic structure described above, only the faces of the preceding generation are built upon. The structure is shown in Figure 5. This is a subset of the structure described by Gosper and Moravec, and it possesses a fractal surface. As a mathematical object, the arrangement in Figure 1 is interesting for the fact that it unexpectedly results in a simple object like a cube. The arrangement in Figure 5 is more interesting as a sculptural object, however.

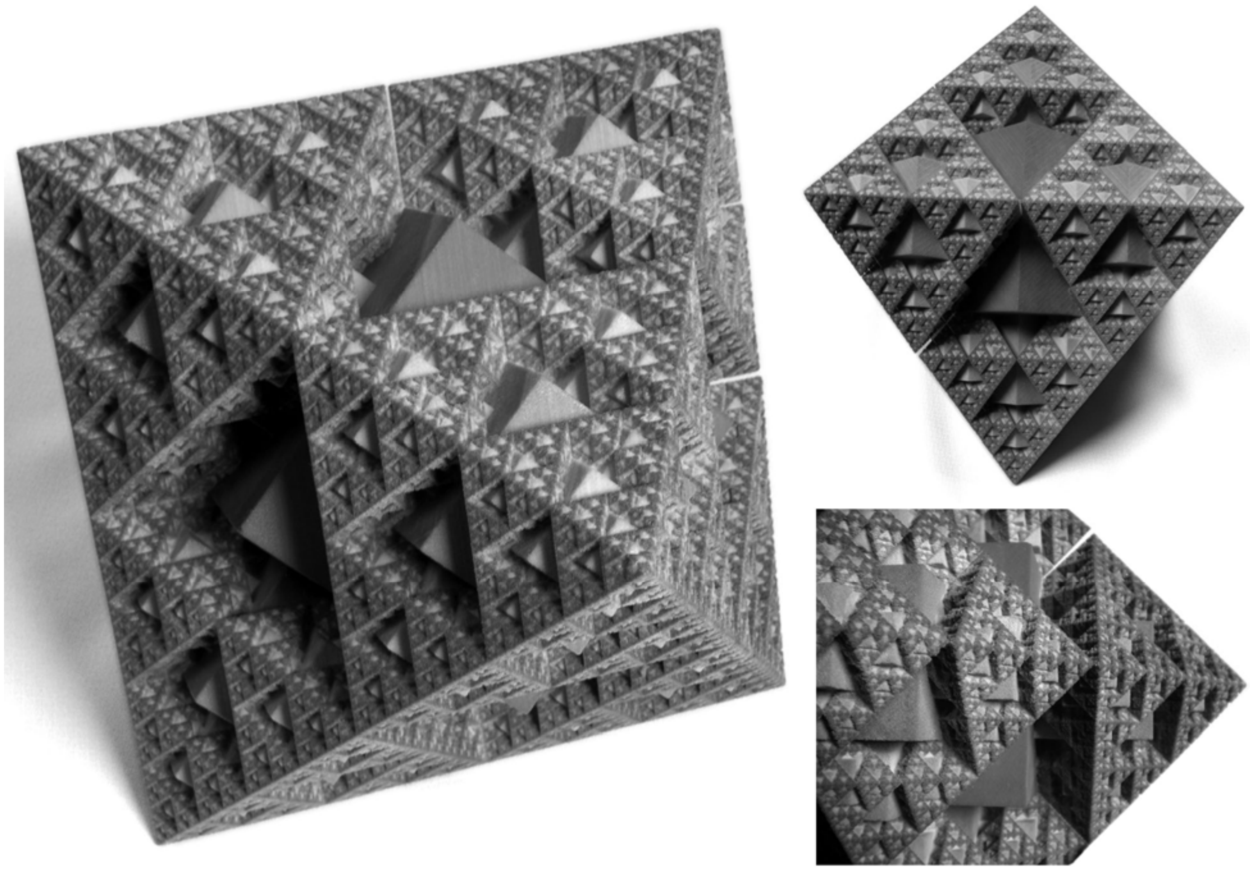

Figure 4: *Three views of the physical fractal crystal.*

Octahedra. An octahedron does possess opposing parallel faces, so all of the octahedra can be oriented the same. A particular orthographic projection of the fractal structure is shown in Figure 6 through four generations. As is the case with the tetrahedra, the convex hull is a cube. Also the same as the tetrahedra case, the footprint of a next-smaller-generation octahedron is the central quarter of an equilateral triangle face. In the octahedra case, however, there is an array of holes in the surface that become infinitesimally small in the limit. There are also fractal cavities within this cube. These features are more readily visible when manipulating a three-dimensional computer model [11]. This arrangement of octahedra was previously described by St. George [12], and a closely related arrangement of octahedra has been described by Goodman-Strauss [13].

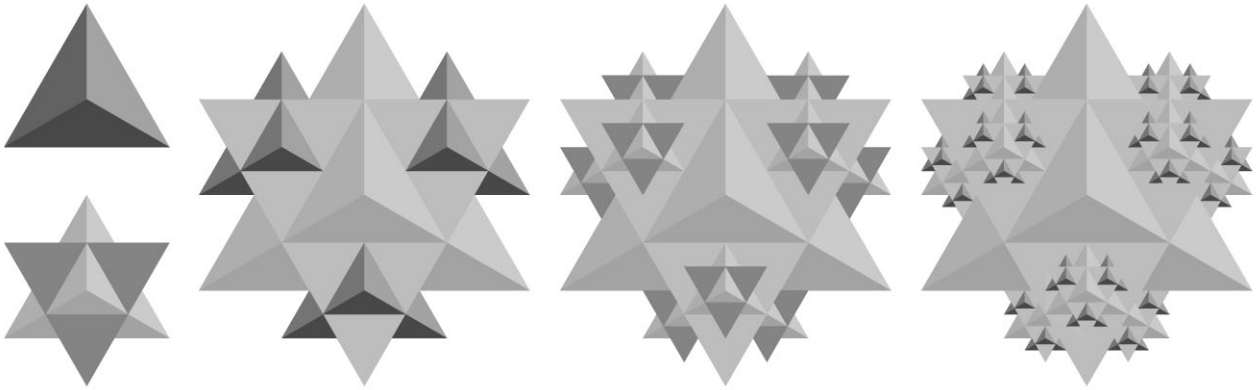

Figure 5: *Stages in the construction of a fractal based on iterative arrangement of successively smaller tetrahedra. New generations are only added on to the preceding generation. In each figure, the earlier generations are lightened in order to bring out the current generation.*

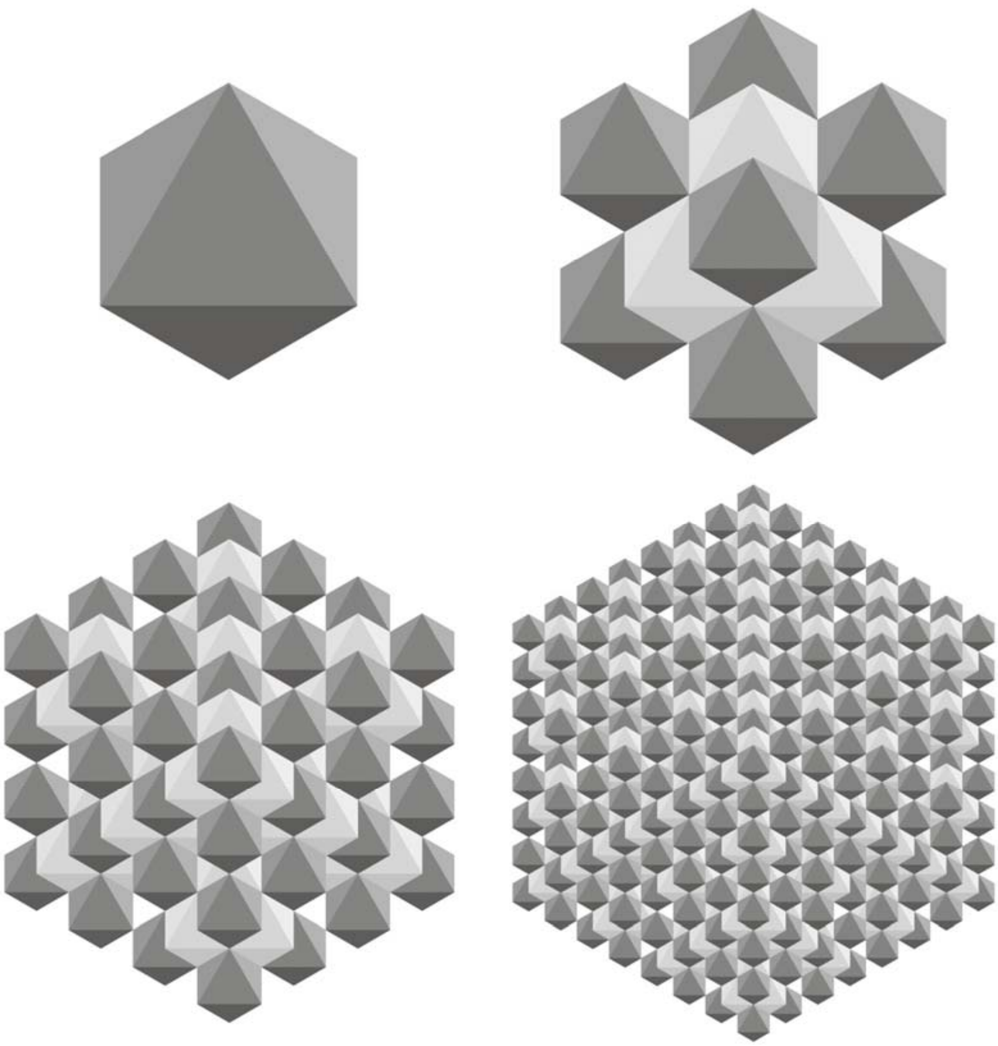

Figure 6: *Stages in the construction of a fractal based on iterative arrangement of successively smaller octahedra. At each stage, the larger octahedra are lightened to emphasize the new generation of octahedra.*

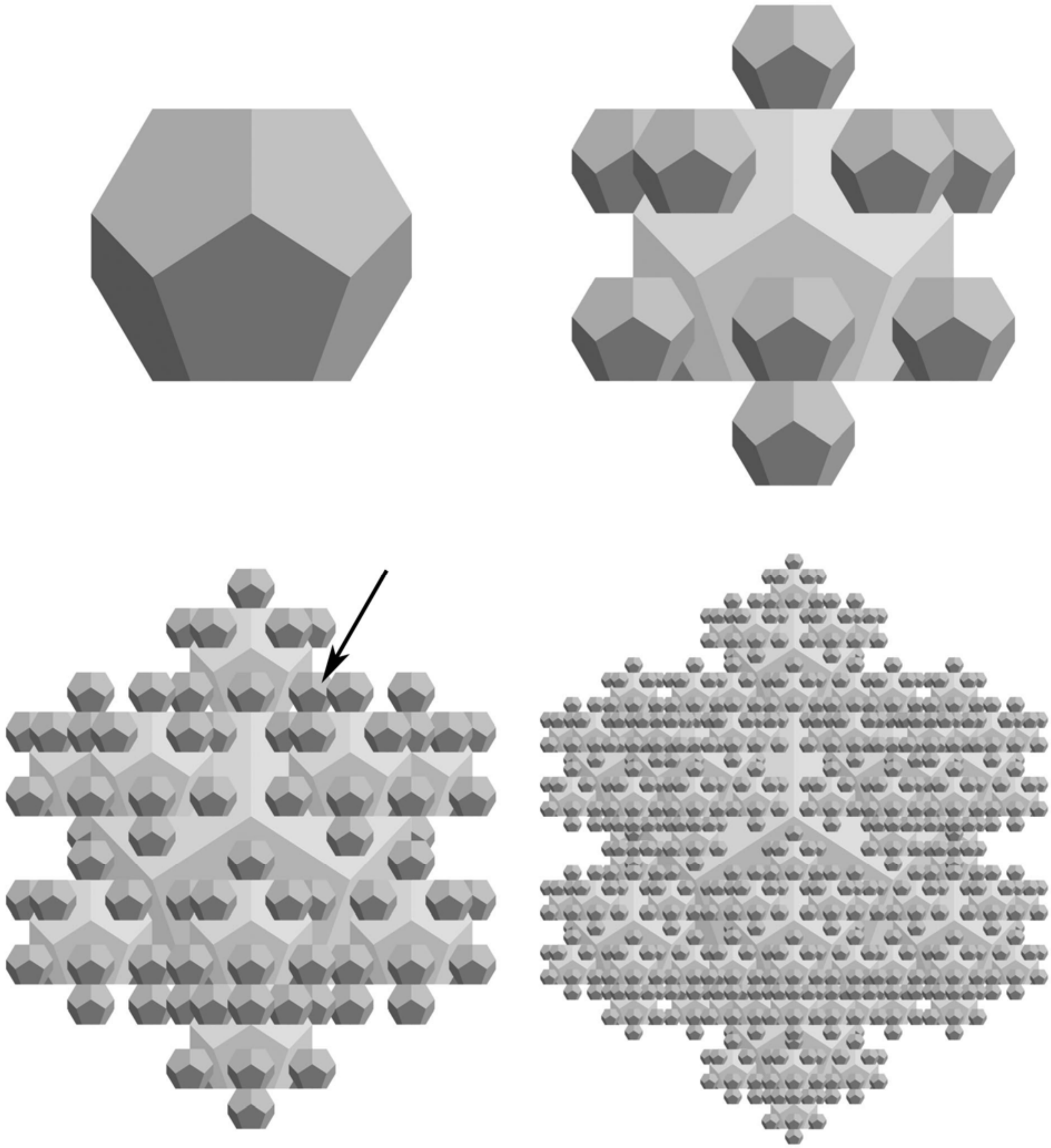

Figure 7: *Stages in the construction of a fractal based on iterative arrangement of successively smaller dodecahedra.*

Dodecahedra. For this case, a scaling factor of 0.5 causes third generation dodecahedra to partially overlap some of the first generation dodecahedra. To avoid this problem a scaling factor of the square of the Golden mean, $(0.618\ldots)^2 \approx 0.382$, was used. With this scaling factor, the third-generation dodecahedra that were a problem now sit flush against the first generation dodecahedra. One of these is indicated by an arrow in Figure 7, which shows a particular orthographic projection through four generations. In this case, a simple geometric convex hull is not seen, giving the fractal a more organic appearance. Closely related fractal arrangements of dodecahedra have been described previously by Goodman-Strauss, who coined the term “dodecafoam” to describe them [14]. The arrangement described here is a subset of dodecafoam. A related arrangement of stellated dodecahedra has been described by David [6].

Conclusions

We have described a fractal arrangement of cubes that has an overall octahedral convex hull with faces that exhibit cavities in the configuration of Sierpinski triangles. These features result from a very simple construction algorithm and were not readily foreseeable at the outset. Analogous fractal structures based on other Platonic Solids have also been described. These objects are interesting both for their mathematical and sculptural properties.

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