Edge-Based Intersected Polyhedral Paper Sculptures Constructed by Interlocking Slitted Planar Pieces

Jace Miller
Computer Science Department
Texas A&M University
jacemiller84@gmail.com

Ergun Akleman
Visualization Department
Texas A&M University
ergun.akleman@gmail.com

Abstract

In this paper, we generalize George Hart's slide-together sculptures as edge-based intersected polyhedral paper sculptures. Edge-based intersected polyhedra are also a conceptual generalization of Kepler's Small Stellated Dodecahedron. These sculptures are constructed by interlocking slitted planar pieces without using glue. We present a simple procedure to construct slitted planar pieces for any given polyhedron. These sculptures can easily be constructed by children and can be used to teach properties of Platonic or Archimedean Solids through hands-on experience.

1 Introduction

George Hart has developed a wide variety of interlocked planar pieces for construction of slide-together sculptures [2] and symmetric modular sculptures [3, 4]. We have observed that George Hart's slide-together sculptures can be generalized and we have developed a procedure that allows people to construct their own slide-together sculptures (See Figure 1). These sculptures consist of a set of interlocked planar pieces (See Figure 2). The planar pieces are interlocked using slits. The resulting sculptures are stable and do not need glue. In our examples, the slitted planar pieces are cut from paper, but other materials such as sheet metal can also be used. The major advantage of our procedure is that it does not require any propriety software and it can easily be taught in a classroom. Our procedure generalizes slide-together sculptures. To generalize other modular sculptures of George Hart [3, 4], we believe that there is a need for software development.

Figure 1: *Our version of Small Stellated Dodecahedron that is constructed by interlocking slitted planar shapes shown in Figure 2.*

There are two parameters that define the shape of the edge-based intersected polyhedral paper sculptures:

1. *Underlying polyhedral structure:* The planar pieces are interlocked through the edge slits that are located in the edges of an underlying polyhedron. There are two conditions for underlying polyhedron to satisfy.
   - *Planar Faces* The faces of polyhedron have to be planar (or developable) since we use paper or sheet metal.

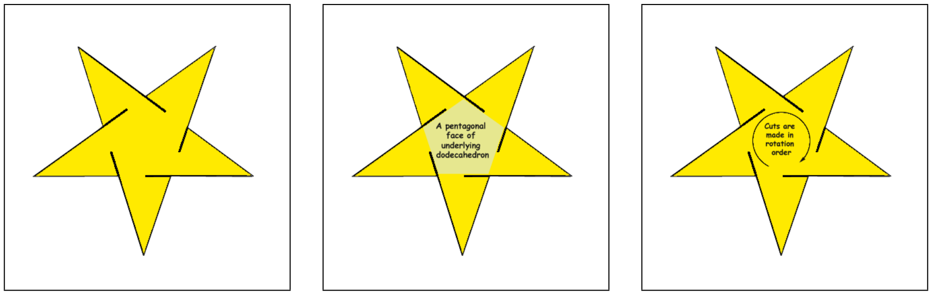

Figure 2: *The shape of slitted planar pieces that are used to construct the intersected sculpture shown in Figure 1. Dark line shows the cuts. Note that cuts are made along with rotation order and each cut goes up to the middle of each edge of the regular pentagon.*

- *No Self-Intersection* The polyhedron must be simple, i.e. the faces must not self intersect. In other words, the stellated polyhedra such as Great Stellated Dodecahedrom or Great Icasahedron cannot be used as underlying polyhedron.

The underlying polyhedron can be non-regular or non-convex as far as these two conditions are satisfied.

2. *The shape of slitted planar pieces.* There is no specific requirement for the shapes of these slitted planar pieces. It is possible to use any shape as far as there is no intersection of planar pieces other than slitted-regions at the edges of underlying polyhedron. On the other hand, the slit positions, orientations and lengths are functions of underlying polyhedron.

We observed that the resulting paper sculptures are conceptually related to first stellation of polyhedra such as Small Stellated Dodecahedron [8]. It is possible to construct an edge-based intersected polyhedral paper sculpture version of first stellation of any polyhedra. Figure 1 shows our version of Small Stellated Dodecahedron in which the underlying polyhedral structure is a regular dodecahedron and the shape of slitted planar pieces are pentagonal stars as shown in Figure 2.

Figure 3: *The circular slitted planar pieces that are used to construct the intersected sculpture shown in Figure 4. Dark lines show the cuts. Note that cuts are made along with rotation order and each cut goes up to the middle of each edge of the regular pentagon.*

If we change the shape of slitted planar pieces from a star to a circle with the same underlying dodecahedron as shown in Figure 3, we get a different looking intersected sculpture as shown in Figure 4.

Figure 4: *A sculpture that is constructed by interlocking circular planar shapes shown in Figure 3. Underlying polyhedron is a dodecahedron, which is the same as the underlying polyhedron of Small Stellated Dodecahedron sculpture in Figure 1.*

2 The Procedure

The procedure to create slitted planar pieces for any given underlying polyhedra is as follows:

1. *Unfold the given underlying polyhedron:* Unfolding is easy for Archimedean and Platonic Solids. For general planar polyhedra, it is possible to unfold the polyhedra using Pepakura, a commercially available polygonal unfolding software [6]. See Figure 5A for an unfolded version of a cube.
2. *Separate all faces of unfolded polyhedra:* When you separate the faces, keep track of the neighborhood relations. Figure 5B shows separated square faces of a cube.
3. *Select a subset of each edge:* Note that in a polyhedron every edge is shared by two faces. In other words, if we unfold a polyhedron and separate all the faces, each edge appears twice[1]. On a per edge basis, a subset of each edge must be selected to match its corresponding edge with which it will interlock. We call the subset of an edge sub-edge. The Figure 5C shows selected sub-edges, that are drawn as a thicker line in "light gray" color.
4. *For every face, draw a closed shape that includes all sub-edges but does not include the rest of the edges:* The Figure 5D shows boundary of closed shapes, which are drawn with a thick "dark gray" pen. Here, the shapes are completely arbitrary. The main rule is that they have to include only sub-edge parts of the edges. In addition, when a closed shape is drawn, care must be taken that it does not extend so far outwards that it would intersect other copies of the shape. Then, draw a consistent rotation inside every closed shape, which are shown as circular arrows in the Figure 5E.
5. *Cut the closed shapes and using the rotation order slit each sub-edge up to its middle point:* The Figure 5E shows the slitted planar pieces in "light gray" color, cuts and slits are drawn with a thick "black" pen. These slitted planar pieces can be used to construct the intersected polyhedron that uses the cube as an underlying polyhedron.
6. *Construct the sculpture by connecting every two corresponding sub-edges:* Connection is simply created by sliding two corresponding sub-edges into each other's slits. The Figure 5F shows a stage in the progress of constructing sculpture. Note that here the pieces are still flat; this is unfolded version of intersected sculpture; when we continue to connect, the shape folds.

The procedure above is general and can work for any given underlying polyhedron. For cutting slits, the only requirement is that the rotation has to be consistent, either counter clockwise or clockwise for all faces

[1]Formally, each edge-face pair is called a half-edge[5] and for every edge of a polyhedron, there are two half-edges. For the sake of simplicity, we call each half-edge an edge and two corresponding half-edges as edges that correspond each other.

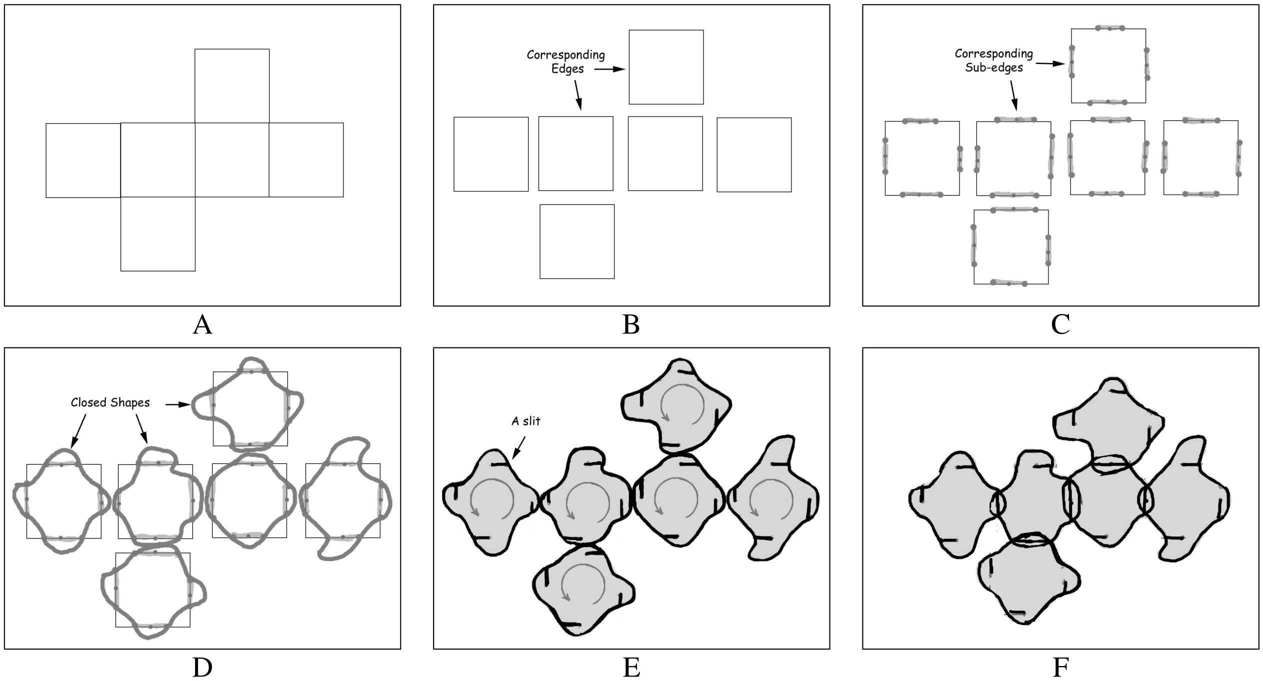

Figure 5: *The procedure to create intersected polyhedral sculptures. In this example, the underlying polyhedron is a cube. Figure 8 shows an intersected sculpture that uses a cube as the underlying polyhedron.*

of underlying polyhedra. The Figure 5F shows clockwise rotation order for all faces. Since we can always have such a consistent rotation for any polyhedral shape [1, 5], any polyhedron can be used as the underlying polyhedron. However, for most people it can be preferable to use Platonic or Archimedean Solids [8] as underlying polyhedra because of their symmetry features. In fact, George Hart's slide-together sculptures uses Platonic Solids, Archimedean Solids and Archimedean Duals as underlying polyhedron [2]. Moreover, for such regular polyhedra unfolded versions are available and therefore there is no need for unfolding software.

Each slitted planar piece can have any shape as shown in Figure 5, but again it can also be desirable to use the same simple shape such as a circle for all slitted planar pieces. For instance, Figures 6 and 7 shows two intersected sculptures that are created using the same circular slitted planar pieces. In Figure 6, the underlying polyhedron is a tetrahedron, in Figure 7, the underlying polyhedron is an octahedron. The same circular slitted pieces can also be used to create a shape that uses an icosahedron as the underlying polyhedron. In other words, it is also possible to design slitted planar pieces that can provide a wide variety of shapes.

Note that in the procedure we suggested to use a subset of edge. On the other hand, in these circular pieces each sub-edge is identical to the original edge. This was our early design and we realized that in such cases, connecting the last edges can be hard requiring us to bend the paper. Therefore, for such cases, the paper has to be thin and flexible. The problem is that thin paper eventually loses its shape. On the other hand, when we use a smaller subset of edges as shown in Figure 8, it is possible to use much harder and thicker paper board. In this way, the construction of the sculpture becomes easier without a need to bend the paper.

We also think that when we use a smaller subset of edge and sub-edge, the resulting sculptures look more interesting since we can see inside of the sculpture. For instance, in Figure 8, the shape is interesting despite the uninteresting simplicity of a cube as the underlying polyhedron.

Using the same slitted planar pieces may also allow the construction of more complicated and growing shapes. For instance, using the same slitted planar pieces that we use to create the sculpture in Figure 8 can be used to create sculptures that uses a subset of $(4,6)$ infinite polyhedron as underlying polyhedron (see Figure 9). Of course, subsets of other infinite polyhedra can also be used as underlying polyhedra.

Note that in Step 4, the closed shapes must be drawn such that they do not extend so far outwards that they would intersect with each other. If the underlying polyhedra is a regular symmetric polyhedra such as a regular dodecahedron, this can be achieved using the stellation diagram of the underlying polyhedron, as explained in [3]. To avoid such collisions for general polyhedra, there is a need for the development of a software that detects intersections.

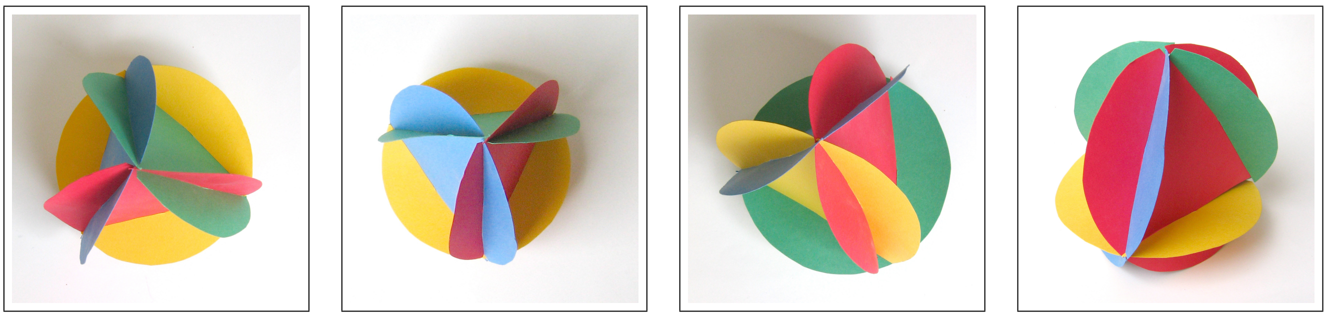

Figure 6: *Circular slitted planar pieces with a tetrahedral underlying polyhedron.*

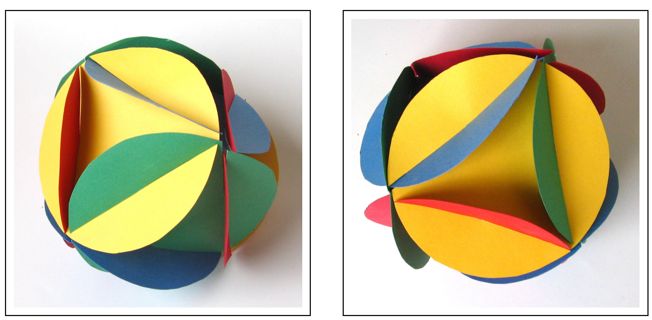

Figure 7: *Octahedral underlying polyhedron with the same circular slitted planar pieces in Figure 6.*

Figure 8: *Cubical underlying polyhedron.*

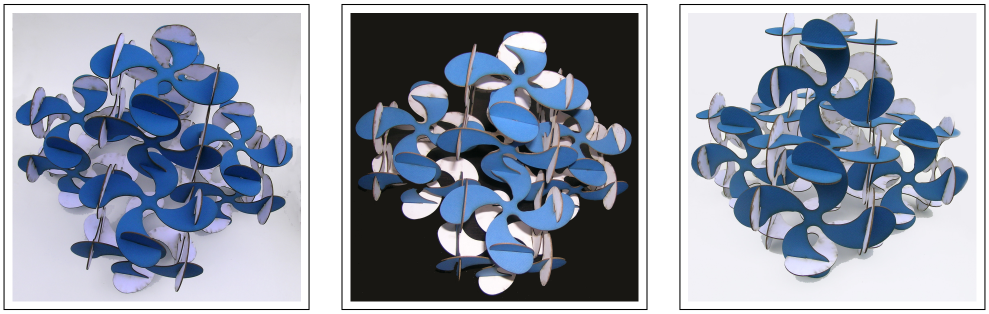

Figure 9: *A subset of infinite polyhedra as underlying polyhedron with the same slitted planar pieces in Figure 8.*

3 Conclusion and Future Work

In this paper, we have introduced a method to construct interesting polyhedral sculptures from paper. These sculptures can be used in classrooms to teach mathematical properties of polyhedral shapes with hands-on artistic experiences. If slitted planar pieces are prepared by teachers, even primary school students can easily construct these sculptures by interlocking these slitted pieces without using glue as also shown by George Hart [2]. We think that more advanced students will love to create their own slitted pieces for Platonic or Archimedean Solids. Students who are interested in computers can go further and use a free modeling software such as TopMod3D [7, 1] to model their own planar polyhedral models, unfold them using Pepakura [6] and create their own sculptures from unfolded polyhedra.

Since these edge-based intersected polyhedral paper sculptures are conceptually related to first stellation of polyhedra such as Kepler's Small Stellated Dodecahedron, it is interesting to ask if there exists similar extensions for second and third stellations to include other Kepler and Poinsot solids. For second and third stellations, it is harder to identify and avoid collisions. For such cases, there is definitely a need for a specialized software. This problem is hard even for regular underlying polyhedra and in fact George Hart is using a software he has developed to create his modular symmetric sculptures [3, 4]. We are planning to develop a software that helps users to create generalized versions of George Hart's modular sculptures by allowing second and third stellations of generalized underlying polyhedra.

We are thankful to George Hart and anonymous reviewers for their extremely helpful suggestions.

References

[1] Akleman, E., Chen, J.: Guaranteeing the 2-manifold property for meshes with doubly linked face list. International Journal of Shape Modeling **5**(2), 149–177 (1999)

[2] Hart, G.: Slide-Together; Geometric Paper Constructions. Teachers' workshop at Bridges 2004, http://www.georgehart.com/slide-togethers/slide-togethers.html

[3] Hart, G.: Symmetric Sculpture. Journal of Mathematics and the Arts, vol 1, no. 1, pp 21-28, March, 2007.

[4] Hart, G.: Modular Kirigami. Proceedings of Bridges Donostia, San Sebastian Spain, 2007, pp. 1-8.

[5] Mantyla, M.: An Introduction to Solid Modeling. Computer Science Press, Rockville, Ma. (1988)

[6] "Pepakura Designer, Official Website" http://www.tamasoft.co.jp/pepakura-en/

[7] "TopMod3D, Official Website" http://www.topmod3d.org

[8] Williams, R.: The Geometrical Foundation of Natural Structures. Dover Publications, Inc. (1972)